MODULI SPACES OF ALGEBRAS OVER NON-SYMMETRIC OPERADS

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ABSTRACT. In this paper we study spaces of algebras over an operad (non-symmetric) in symmetric monoidal model categories. We begin by extending a result of Rezk for simplicial sets and simplicial modules to a general framework. We then apply this result to the construction of moduli stacks of algebras over an operad in a homotopical algebraic geometry context in the sense of Toën and Vezzosi. We show that the stack of unital A-infinity algebras with underlying perfect complex concentrated in a fixed interval is a formal Zariski open subset of the corresponding stack of non-necessarily unital A-infinity algebras. The analogue for associative algebra structures on a finite-dimensional vector space was noticed by Gabriel.

INTRODUCTION

Let $\Bbbk$ be a commutative ring. An associative algebra structure on a free $\Bbbk$-module F of rank n with basis $\{e_1, \dots, e_n\} \subset F \cong \Bbbk^n$ is determined by the structure constants c_{ij}^k, $1 \leq i, j, k \leq n$, such that

$$e_i \cdot e_j = \sum_{k=1}^n c_{ij}^k e_k.$$

The associativity condition $(e_i \cdot e_j) \cdot e_k = e_i \cdot (e_j \cdot e_k)$ is equivalent to the following identities of structure constants,

$$\sum_{m=1}^n c_{ij}^m c_{mk}^l = \sum_{m=1}^n c_{im}^l c_{jk}^m, \qquad 1 \leq l \leq n.$$

Therefore, the moduli space of associative algebra structures on a free module of rank n is the affine subspace $\operatorname{Spec} R \subset \mathbb{A}^{n^3}$,

$$R = \Bbbk[c_{ij}^k \,;\, 1 \leq i, j, k \leq n] \left/ \left(\sum_{m=1}^n \left(c_{ij}^m c_{mk}^l - c_{im}^l c_{jk}^m \right) \,;\, 1 \leq i, j, k, l \leq n \right) \right. .$$

A unital associative algebra structure on F is given by an associative algebra structure together with a unit element

$$\mathbf{1} = \sum_{i=1}^n a_i e_i \in F$$

---

satisfying the following equations, $1 \leq k \leq n$, which are equivalent to $\mathbf{1} \cdot e_i = e_i = e_i \cdot \mathbf{1}$, $1 \leq i \leq n$,

$$(a_1, \ldots, a_n) \begin{pmatrix} c_{11}^k & \cdots & c_{1n}^k \\ \vdots & \ddots & \vdots \\ c_{n1}^k & \cdots & c_{nn}^k \end{pmatrix} = (0, \ldots, \overset{\overset{k^{\text{th}} \text{ place}}{\downarrow}}{1}, \ldots, 0),$$

$$\begin{pmatrix} c_{11}^k & \cdots & c_{1n}^k \\ \vdots & \ddots & \vdots \\ c_{n1}^k & \cdots & c_{nn}^k \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow k^{\text{th}} \text{ place}.$$

Hence, the moduli space of unital associative algebra structures on a free module of rank n is the affine subspace $\operatorname{Spec} S \subset \mathbb{A}^{n^3+n}$,

$$S = R[a_1, \ldots, a_n] \Bigg/ \left(\delta_{jk} - \sum_{i=1}^n a_i c_{ij}^k, \ \delta_{jk} - \sum_{i=1}^n c_{ji}^k a_i \ ; \ 1 \leq j, k \leq n \right),$$

where $\delta_{kk} = 1$ and $\delta_{jk} = 0$ if $j \neq k$.

The morphism $f \colon \operatorname{Spec} S \to \operatorname{Spec} R$ consisting of forgetting the unit is induced by the inclusion $R \subset S$. This morphism is a categorical monomorphism since an associative algebra may have at most one unit. Moreover, f is an open immersion, compare [Gab74, 2.1 Lemma] and [CB93, page 4].

Associative algebra structures are somewhat rigid. We are rather interested in them up to isomorphism. The algebraic group $\operatorname{GL}_n \cong \operatorname{Aut}_{\Bbbk}(F)$ acts on $\operatorname{Spec} R$. The orbits are the isomorphism classes of associative algebra structures. The isotropy group at a given point is the automorphism group of the corresponding associative algebra structure.

In order to obtain a meaningful quotient which remembers all this, we must move to the category of stacks. The quotient stack

$$\underline{\operatorname{Ass}}_n = \operatorname{Spec} R / \operatorname{GL}_n$$

is the moduli space of associative algebras on rank n vector bundles. The same applies to $\operatorname{Spec} S$, and the quotient

$$\underline{\operatorname{uAss}}_n = \operatorname{Spec} S / \operatorname{GL}_n$$

is the moduli stack of unital associative algebras on rank n vector bundles. The morphism $f \colon \operatorname{Spec} S \to \operatorname{Spec} R$ induces a morphism between these stacks

$$\underline{f} \colon \underline{\operatorname{uAss}}_n \longrightarrow \underline{\operatorname{Ass}}_n$$

which is again an open immersion [LMB00, 5].

Up to equivalence, these stacks can be described as follows. The category $\operatorname{Aff}_{\Bbbk}$ of affine schemes over $\Bbbk$ is opposite to the category of commutative (associative and unital) $\Bbbk$-algebras. We assume this category is endowed with a subcanonical Grothendieck topology. For any commutative $\Bbbk$-algebra R, let $\operatorname{Ass}_n(R)$ be the category of associative R-algebras whose underlying R-module is locally free of

rank n. Denote $i\operatorname{Ass}_n(R)$ the subcategory of isomorphisms. Change of coefficient functors give rise to a functor

$$\operatorname{Aff}_{\Bbbk}^{\mathrm{op}} \longrightarrow \operatorname{Groupoids},$$
$$R \mapsto i\operatorname{Ass}_n(R),$$

which is the functor of points of the stack $\underline{\operatorname{Ass}}_n$. Similarly, if $\operatorname{uAss}_n(R)$ is the subcategory of unital associative R-algebras whose underlying R-module is locally free of rank n, the functor of points of $\underline{\operatorname{uAss}}_n$ is

$$\operatorname{Aff}_{\Bbbk}^{\mathrm{op}} \longrightarrow \operatorname{Groupoids},$$
$$R \mapsto i\operatorname{uAss}_n(R),$$

and the morphism $\underline{f} \colon \underline{\operatorname{uAss}}_n \to \underline{\operatorname{Ass}}_n$ is induced by the inclusion.

In this paper we aim at studying moduli spaces of (unital) associative differential graded (DG) algebras with a given underlying DG-module M, and the map induced by forgetting the unit. In the differential graded world, it is reasonable to replace $\operatorname{Aff}_{\Bbbk}$ with the opposite category of commutative differential graded $\Bbbk$-algebras. Therefore we should work in a homotopical algebraic geometry context, in the sense of Toën and Vezzosi [TV08].

For any commutative DG-algebra R, denote $\operatorname{Ass}_M(R)$ the category of associative DG-algebras whose underlying DG-module is locally quasi-isomorphic to $M \otimes_{\Bbbk} R$ (locally with respect to a model topology on $\operatorname{Aff}_{\Bbbk}$). Let $w\operatorname{Ass}_M(R)$ be the subcategory of quasi-isomorphisms (a.k.a. weak equivalences). We will show that the functor

$$\operatorname{Aff}_{\Bbbk}^{\mathrm{op}} \longrightarrow \operatorname{Spaces},$$
$$R \mapsto |w\operatorname{Ass}_M(R)|,$$

defines a 1-geometric stack $\underline{\operatorname{Ass}}_M$ whenever M is perfect (Theorem 5.6). Here $|\mathscr{C}|$ denotes the classifying space of the category $\mathscr{C}$. Similarly for

$$\operatorname{Aff}_{\Bbbk}^{\mathrm{op}} \longrightarrow \operatorname{Spaces},$$
$$R \mapsto |w\operatorname{uAss}_M(R)|,$$

where $\operatorname{uAss}_M(R)$ denotes the subcategory of unital associative DG-algebras. This functor will define the 1-geometric moduli stack $\underline{\operatorname{uAss}}_M$ of unital associative DG-algebras with underlying object M. Since $\operatorname{Aff}_{\Bbbk}$ is a stable model category in this context, all affine morphisms are flat, hence it would be reasonable to expect the morphism induced by the inclusion

$$\underline{f} \colon \underline{\operatorname{uAss}}_M \longrightarrow \underline{\operatorname{Ass}}_M$$

to be a Zariski open immersion. We will prove that $\underline{f}$ is in fact a formal Zariski open immersion (see Corollary 5.14 for the specific statement in the appropriate context). The adjective 'formal' comes from the fact that $\underline{f}$ is not locally of finite presentation, unlike in the classical case. Actually, neither $\underline{\operatorname{uAss}}_M$ nor $\underline{\operatorname{Ass}}_M$ is locally finitely presented.

To prove these results, we will give an alternative construction of $\underline{\operatorname{Ass}}_M$, $\underline{\operatorname{uAss}}_M$, and $\underline{f}$ as a quotient of a formal Zariski open immersion between affine stacks of algebra structures by the action of the affine group stack $\mathbb{R}\underline{\operatorname{Aut}}_{\Bbbk}(M)$ of derived automorphisms of M.

In order to carry out this strategy we will have to overcome several difficulties of homotopical nature. The best way to study and compare different kinds of algebra structures is by means of operads. Moduli spaces of algebra structures over a given symmetric operad were considered by Rezk for simplicial sets and simplicial modules [Rez96]. These spaces are defined as mapping spaces in a simplicial model category of operads. His results were used by Toën and Vezzosi to show the existence of a moduli stack of algebras over an operad with a given underlying simplicial module in the context of derived algebraic geometry [TV08, 2.2.6.2].

Symmetric operads on a given symmetric monoidal model category $\mathscr{V}$ need not form a model category with weak equivalences defined as in $\mathscr{V}$ [Hin97, Hin03, BM03], and if so, the model structure need not be simplicial, which would be necessary to adapt Rezk's arguments.

Since we are interested in associative algebras, we may choose to work with non-symmetric operads, which do have a transferred model structure $\mathrm{Op}(\mathscr{V})$ [Mur11a, Theorem 1.1] whenever $\mathscr{V}$ satisfies the two usual compatibility axioms between the monoidal and the model structures: the push-out product axiom and the monoid axiom [SS00], and some extra smallness assumptions (see this introduction's last paragraph).

The problem of $\mathrm{Op}(\mathscr{V})$ not being simplicial remains, and we devote most of this paper to develop techniques which avoid the use of a simplicial structure. Instead, we directly work with simplicial resolutions in the sense of Dwyer and Kan [DK80] to construct mapping spaces. For the computation of homotopy types of mapping spaces in $\mathrm{Op}(\mathscr{V})$ we need general conditions under which change of base operad functors induce Quillen equivalences between categories of algebras. Such conditions are studied in [Mur11a], but the criteria therein are not general enough for our intended geometric applications. If we want to place our problem in an honest homotopical algebraic geometry context, unless $\Bbbk$ is a $\mathbb{Q}$-algebra we must move from differential graded algebra to stable homotopy. More precisely, we must replace DG-modules over $\Bbbk$ with modules over the Eilenberg–MacLane ring spectrum $H\Bbbk$. The symmetric monoidal model category of $H\Bbbk$-modules is Quillen equivalent to that of DG-modules over $\Bbbk$, and the former is better behaved than the later, except from the fact that $H\Bbbk$ is not cofibrant as an $H\Bbbk$-module. This is why we need to prove [Mur11a, Theorem 1.3] under weaker assumptions (Theorem 3.11).

The fact that $\underline{f}\colon \underline{\mathrm{uAss}}_M \to \underline{\mathrm{Ass}}_M$ is a formal Zariski open immersion (Corollary 5.14) will be obtained as a consequence of [Mur11b].

Our techniques are general enough to define moduli stacks of algebras over any non-symmetric operad. Even to consider saces of algebras living in a different (not necessarily symmetric) monoidal model category $\mathscr{C}$ appropriately enriched over $\mathscr{V}$ (i.e. a $\mathscr{V}$-algebra) satisfying the non-symmetric analogues of the two previous compatibility axioms [Mur11a]. This allows to extend some of the aforementioned results to DG-categories.

The paper is structured as follows. Section 1 is a reminder on operads and their algebras, we introduce some constructions and notation that will be used throughout the paper. Section 2 deals with endomorphism operads. They are not functorial in the argument but satisfy some homotopy invariance properties. In Section 3 we prove a new version of [Mur11a, Theorem 1.3] under weaker hypotheses. This theorem studies when a weak equivalence of operads induces a Quillen equivalence

between their model categories of algebras. In Section 4 we compute the homotopy fiber of the functor which sends an algebra over an operad to the underlying object. Finally, we present geometric constructions in Section 5.

We assume the reader familiarity with abstract homotopy theory. All we need in this paper is covered by standard references, such as [Hov99, Hir03]. When we talk about spaces we always mean simplicial set. We endow the category of bisimplicial sets with the Moerdijk model structure, where fibrations and weak equivalences are detected by the diagonal. Any bisimplicial set will be regarded as a simplicial set through the diagonal functor, and any object in a given category will be regarded as a constant simplicial object when needed. See [GJ99] for simplicial homotopy theory. For the last section, the reader should also be familiar with homotopical algebraic geometry [TV05, TV08].

When taking nerves of 'big' categories, i.e. categories with a 'class' of objects, we assume with no explicit mention that we are working with universes so that everything makes sense from a set-theoretical point of view in a larger universe.

The tensor product in monoidal categories is denoted by $\otimes$ and its unit object by $\mathbb{I}$. If we need to emphasize the category we are talking about we will add it as a subscript, e.g. $\otimes_{\mathscr{V}}$ and $\mathbb{I}_{\mathscr{V}}$. Internal morphism objects in $\mathscr{V}$ will always be denoted by $\mathrm{Hom}_{\mathscr{V}}$.

Extra homotopical standing assumptions on $\mathscr{V}$ and $\mathscr{C}$ will be the following: these model categories are cofibrantly generated and they have sets of generating cofibrations and generating trivial cofibrations with presentable sources. These conditions are fulfilled if $\mathscr{V}$ and $\mathscr{C}$ are combinatorial.

1. Operads and algebras

This section contains some background about operads and their algebras.

Definition 1.1. A *(non-symmetric) operad* $\mathcal{O}$ in $\mathscr{V}$ is a sequence of objects $\mathcal{O}(n)$, $n \geq 0$, together with composition laws

$$\circ_i \colon \mathcal{O}(m) \otimes \mathcal{O}(n) \longrightarrow \mathcal{O}(m + n - 1), \qquad 1 \leq i \leq m, \quad n \geq 0,$$

and a unit morphism $u \colon \mathbb{I}_{\mathscr{V}} \to \mathcal{O}(1)$ satisfying some relations. For instance, if $\mathscr{V}$ is the category of modules over a commutative ring, the relations are:

- $(a \circ_i b) \circ_j c = (a \circ_j c) \circ_{i+n-1} b$ if $1 \leq j < i$ and $c \in \mathcal{O}(n)$.
- $(a \circ_i b) \circ_j c = a \circ_i (b \circ_{j-i+1} c)$ if $b \in \mathcal{O}(m)$ and $i \leq j < m + i$.
- $u \circ_1 a = a$.
- $a \circ_i u = a$.

These equations can be translated into commutative diagrams which describe the relations for arbitrary $\mathscr{V}$, see [Mur11a, Remark 2.6].

An *operad morphism* $\phi \colon \mathcal{O} \to \mathcal{P}$ is a sequence of morphisms $\phi(n) \colon \mathcal{O}(n) \to \mathcal{P}(n)$ in $\mathscr{V}$, $n \geq 0$, compatible with the composition laws, e.g. in the category of modules

over a commutative ring,

$$\phi(m + n - 1)(a \circ_i b) = \phi(m)(a) \circ_i \phi(n)(b), \qquad \phi(1)(u) = u.$$

Let $\mathscr{V}^{\mathbb{N}}$ be the product of countably many copies of $\mathscr{V}$, indexed by the non-negative integers, with model structure defined coordinatewise. The forgetful functor

$$\mathrm{Op}(\mathscr{V}) \longrightarrow \mathscr{V}^{\mathbb{N}},$$
$$\mathcal{O} \mapsto \{\mathcal{O}(n)\}_{n \geq 0},$$

has a left adjoint, the *free operad* functor

$$\mathcal{F} \colon \mathscr{V}^{\mathbb{N}} \longrightarrow \mathrm{Op}(\mathscr{V}).$$

This adjoint pair is a Quillen pair. Actually, this is how we define the model structure on $\mathrm{Op}(\mathscr{V})$ [Mur11a].

The $\mathscr{V}$-*algebra* structure on $\mathscr{C}$ is given by a strong braided monoidal functor $\mathscr{V} \to Z(\mathscr{C})$, where $Z(\mathscr{C})$ is the center of $\mathscr{C}$ [JS91]. Such a functor consists of an ordinary functor $z \colon \mathscr{V} \to \mathscr{C}$, preserving (trivial) cofibrations, together with natural isomorphisms,

$$multiplication \colon z(X) \otimes_{\mathscr{C}} z(X') \longrightarrow z(X \otimes_{\mathscr{V}} X'),$$
$$unit \colon \mathbb{I}_{\mathscr{C}} \longrightarrow z(\mathbb{I}_{\mathscr{V}}),$$
$$\zeta(X, Y) \colon z(X) \otimes_{\mathscr{C}} Y \longrightarrow Y \otimes_{\mathscr{C}} z(X),$$

satisfying some coherence laws, see [Bor94, Definition 6.4.1] and [Mur11a, §7]. Moreover, the functor $z(-) \otimes Y \colon \mathscr{V} \to \mathscr{C}$ must have a right adjoint for any Y in $\mathscr{C}$,

$$\mathrm{Hom}_{\mathscr{C}}(Y, -) \colon \mathscr{C} \longrightarrow \mathscr{V}.$$

These morphism objects in $\mathscr{V}$ define a tensored $\mathscr{V}$-enrichment of $\mathscr{C}$ over $\mathscr{V}$, see [JK02, Appendix]. Furthermore, since z is a functor to the center, this enrichment can be enhanced to a monoidal $\mathscr{V}$-category structure on $\mathscr{C}$. For this, given two objects Y and Y' in $\mathscr{C}$, we define the *evaluation morphism*

$$\mathrm{evaluation} \colon z(\mathrm{Hom}_{\mathscr{C}}(Y, Y')) \otimes Y \longrightarrow Y',$$

as the adjoint of the identity in $\mathrm{Hom}_{\mathscr{C}}(Y, Y')$, and given two other objects X and X' in $\mathscr{C}$ we definine the morphism in $\mathscr{V}$

$$\otimes_{\mathscr{C}} \colon \mathrm{Hom}_{\mathscr{C}}(X, X') \otimes_{\mathscr{V}} \mathrm{Hom}_{\mathscr{C}}(Y, Y') \longrightarrow \mathrm{Hom}_{\mathscr{C}}(X \otimes_{\mathscr{C}} Y, X' \otimes_{\mathscr{C}} Y')$$

as the adjoint of

$$z(\mathrm{Hom}_{\mathscr{C}}(X, X') \otimes_{\mathscr{V}} \mathrm{Hom}_{\mathscr{C}}(Y, Y')) \otimes_{\mathscr{C}} X \otimes_{\mathscr{C}} Y$$

$$\Big\downarrow {\scriptstyle \mathrm{multiplication}^{-1} \otimes_{\mathscr{C}} \mathrm{id}_{X \otimes_{\mathscr{C}} Y}} \quad {\scriptstyle \cong}$$

$$z(\mathrm{Hom}_{\mathscr{C}}(X, X')) \otimes_{\mathscr{C}} z(\mathrm{Hom}_{\mathscr{C}}(Y, Y')) \otimes_{\mathscr{C}} X \otimes_{\mathscr{C}} Y$$

$$\Big\downarrow {\scriptstyle \zeta(\mathrm{Hom}_{\mathscr{C}}(Y, Y'), X)} \quad {\scriptstyle \cong}$$

$$z(\mathrm{Hom}_{\mathscr{C}}(X, X')) \otimes_{\mathscr{C}} X \otimes_{\mathscr{C}} z(\mathrm{Hom}_{\mathscr{C}}(Y, Y')) \otimes_{\mathscr{C}} Y$$

$$\Big\downarrow {\scriptstyle \mathrm{evaluation} \otimes_{\mathscr{C}} \mathrm{evaluation}}$$

$$X' \otimes_{\mathscr{C}} Y'$$

Definition 1.2. Let $\mathcal{O}$ be an operad in $\mathscr{V}$. An $\mathcal{O}$-algebra A in $\mathscr{C}$ is an object of $\mathscr{C}$ equipped with structure morphisms

$$\nu_n \colon z(\mathcal{O}(n)) \otimes A^{\otimes n} \longrightarrow A, \qquad n \geq 0,$$

satisfying some compatibility relations with the composition laws and the unit of $\mathcal{O}$. If $\mathscr{V} = \mathscr{C}$ is the category of modules over a commutative ring, z is the identity functor, and we denote $\nu_n(a \otimes c_1 \otimes \cdots \otimes c_n) = a(c_1, \ldots, c_n)$, then the relations are:

- $(a \circ_i b)(c_1, \ldots, c_{m+n-1}) = a(c_1, \ldots, c_{i-1}, b(c_i, \ldots, c_{i+n-1}), c_{i+n}, \ldots, c_{m+n-1})$ if $a \in \mathcal{O}(m)$ and $b \in \mathcal{O}(n)$.
- $u(c) = c$.

These relations can be translated into commutative diagrams describing the relations for arbitrary $\mathscr{C}$, see [Mur11a, Definition 7.1].

An $\mathcal{O}$-*algebra morphism* $f \colon A \to B$ is a morphism in $\mathscr{C}$ compatible with the structure morphisms. In the module case the compatibility condition is

$$f(a(c_1, \ldots, c_n)) = a(f(c_1), \ldots, f(c_n)).$$

In the general case, the obvious squares must commute, see again [Mur11a, Definition 7.1].

Given $\mathcal{O}$ in $\mathrm{Op}(\mathscr{V})$, the forgetful functor

$$\mathrm{Alg}_{\mathscr{C}}(\mathcal{O}) \longrightarrow \mathscr{C}$$

has a left adjoint, the *free $\mathcal{O}$-algebra* functor

$$\mathcal{F}_{\mathcal{O}} \colon \mathscr{C} \longrightarrow \mathrm{Alg}_{\mathscr{C}}(\mathcal{O}).$$

This adjoint pair is a Quillen pair. Actually, this is how we define the model structure on $\mathrm{Alg}_{\mathscr{C}}(\mathcal{O})$ [Mur11a, §8].

Algebras over an operad can be alternatively described by means of the endomorphism operad.

Definition 1.3. The *endomorphism operad* of an object Y in $\mathscr{C}$ is the operad $\mathtt{End}_{\mathscr{C}}(Y)$ in $\mathscr{V}$ with

$$\mathtt{End}_{\mathscr{C}}(Y)(n) = \mathrm{Hom}_{\mathscr{C}}(Y^{\otimes n}, Y).$$

The unit is the identity in Y. For module categories, the composition laws of this operad are given by composition of multilinear maps, as in the first equation of the previous definition. The definition in the general case is a translation to the language of commutative diagrams, see [Mur11a, Definition 7.1].

Lemma 1.4. *For any operad $\mathcal{O}$ in $\mathscr{V}$ and any object Y in $\mathscr{C}$, there is a bijection between the morphisms $\mathcal{O} \to \mathtt{End}_{\mathscr{C}}(Y)$ in $\mathrm{Op}(\mathscr{V})$ and the $\mathcal{O}$-algebra structures on Y.*

Proof. The adjoint of the morphism $\phi(n) \colon \mathcal{O}(n) \to \mathtt{End}_{\mathscr{C}}(Y)(n) = \mathrm{Hom}_{\mathscr{C}}(Y^{\otimes n}, Y)$ is the structure morphism $\nu_n \colon z(\mathcal{O}(n)) \otimes Y^{\otimes n} \to Y$. $\qquad \square$

In the last section, when identifying the tangent complex of the moduli stack of algebras over an operad, we will need the notion of module over an algebra over an operad that we now recall.

Definition 1.5. Let $\mathcal{O}$ be an operad in $\mathscr{V}$ and A an $\mathcal{O}$-algebra in $\mathscr{C}$. An A-*module* is an object M in $\mathscr{C}$ together with morphisms

$$\nu_{n,i} \colon z(\mathcal{O}(n)) \otimes A^{\otimes(i-1)} \otimes M \otimes A^{\otimes(n-i)} \longrightarrow A, \qquad n \geq 1, \quad 1 \leq i \leq n,$$

satisfying certain associativity laws. If $\mathscr{C}$ has a zero object these laws can be encoded as follows: the coproduct $A \coprod M$ in $\mathscr{C}$ has an $\mathcal{O}$-algebra structure, called *semidirect product* and denoted by $A \ltimes M$,

$$
z(\mathcal{O}(n)) \otimes \left(A \coprod M \right)^{\otimes n} \xrightarrow{\hspace{5cm}} A \coprod M
$$

$$
\parallel
$$

$$
z(\mathcal{O}(n)) \otimes A^{\otimes n} \coprod \left(\coprod_{i=1}^{n} z(\mathcal{O}(n)) \otimes A^{\otimes(i-1)} \otimes M \otimes A^{\otimes(n-i)} \right) \coprod \cdots
$$

given by the structure morphisms of the $\mathcal{O}$-algebra A on $z(\mathcal{O}(n)) \otimes A^{\otimes n}$, $n \geq 0$, the morphisms $\nu_{n,i}$ on $z(\mathcal{O}(n)) \otimes A^{\otimes(i-1)} \otimes M \otimes A^{\otimes(n-i)}$, $1 \leq i \leq n$, and the zero morphism on the rest of factors. The semidirect product $A \ltimes M$ is also called *square zero extension* of A by M.

Morphisms of A-modules are morphisms in $\mathscr{C}$ compatible with the $\nu_{n,i}$'s.

Remark 1.6. If $\mathcal{O}$ is the operad $\mathtt{Ass}$, whose algebras are monoids in $\mathscr{C}$, then an A-module is the same as a bimodule in the usual sense. For any $\mathcal{O}$, the $\mathcal{O}$-algebra A is itself an A-module with $\nu_{n,i} = \nu_n$ for all $1 \leq i \leq n$. Moreover, if M is an A-module and N is an object in $\mathscr{V}$ then $M \otimes z(N)$ is an A-module with $\nu_{n,i}^{M \otimes z(N)} = (\nu_{n,i}^{M} \otimes z(N))\zeta(N, A^{\otimes(n-i)})$.

Algebra morphisms can also be described by means of endomorphism operads in diagram categories.

Given a small category I, the category of I-*shaped diagrams* $\mathscr{C}^I$ is the category of functors $I \to \mathscr{C}$ and natural transformations between them. It is complete and cocomplete, and inherits from $\mathscr{C}$ a biclosed monoidal structure. The tensor product of two diagrams $Y, Y' \colon I \to \mathscr{C}$ is defined as

$$
(Y \otimes Y')(i) = Y(i) \otimes Y'(i), \qquad i \in I.
$$

This category comes also equipped with a functor from $\mathscr{V}$ to its center,

$$
z^I \colon \mathscr{V} \longrightarrow Z(\mathscr{C}^I), \qquad z^I(X)(i) = z(X), \qquad \zeta^I(X,Y)(i) = \zeta(X, Y(i)).
$$

The right adjoint of $z^I(-) \otimes Y \colon \mathscr{V} \to \mathscr{C}^I$ is the functor

$$
\mathrm{Hom}_{\mathscr{C}^I}(Y, -) \colon \mathscr{C}^I \longrightarrow \mathscr{V}
$$

defined by the following end in $\mathscr{V}$ [Mac98, IX.5],

$$
\mathrm{Hom}_{\mathscr{C}^I}(Y, Y') = \int_{i \in I} \mathrm{Hom}_{\mathscr{C}}(Y(i), Y'(i)).
$$

The next result follows readily from the previous lemma and the universal property of an end.

Corollary 1.7. *For any operad $\mathcal{O}$ in $\mathscr{V}$, any small category I, and any diagram $Y \colon I \to \mathscr{C}$, there is a bijection between the morphisms $\mathcal{O} \to \mathrm{End}_{\mathscr{C}^I}(Y)$ in $\mathrm{Op}(\mathscr{V})$ and the collections of $\mathcal{O}$-algebra structures on the objects $Y(i)$ in $\mathscr{C}$, $i \in I$, such that the morphisms in the diagram are $\mathcal{O}$-algebra morphisms.*

Any functor $F \colon J \to I$ between small categories induces by precomposition a strong monoidal $\mathscr{V}$-functor $F^* \colon \mathscr{C}^I \longrightarrow \mathscr{C}^J$, $F^*(Y) = YF$. The morphisms

$$
F^*(Y, Z) \colon \mathrm{Hom}_{\mathscr{C}^I}(Y, Z) \longrightarrow \mathrm{Hom}_{\mathscr{C}^J}(YF, ZF)
$$

are defined by the universal property of an end. In particular F gives rise to morphisms between endomorphism operads $F^* \colon \mathrm{End}_{\mathscr{C}^I}(Y) \to \mathrm{End}_{\mathscr{C}^J}(YF)$ in $\mathrm{Op}(\mathscr{V})$.

Denote Δ the simplex category, whose objects are the finite ordinals
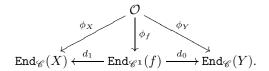
$$\mathbf{n} = \{0 < \cdots < n\}, \qquad n \geq 0,$$

and morphisms are non-decreasing maps. These ordinals are regarded as categories with morphisms going upwards $i \to j$, $i \leq j$. The category Δ is generated by the coface and codegeneracy maps,

$$d^i \colon \mathbf{n} - \mathbf{1} \longrightarrow \mathbf{n}, \qquad s^i \colon \mathbf{n} + \mathbf{1} \longrightarrow \mathbf{n}, \qquad 0 \leq i \leq n,$$

subject to the duals of the usual simplicial relations.

A morphism in $\mathscr{C}$ is the same as a functor $\mathbf{1} \to \mathscr{C}$. Applying the previous corollary to $I = \mathbf{1}$, we deduce the following characterization of $\mathcal{O}$-algebra morphisms.

Corollary 1.8. *Given an operad $\mathcal{O}$ in $\mathscr{V}$ and two $\mathcal{O}$-algebras in $\mathscr{C}$, X and Y, defined by morphisms $\phi_X \colon \mathcal{O} \to \mathrm{End}_{\mathscr{C}}(X)$ and $\phi_Y \colon \mathcal{O} \to \mathrm{End}_{\mathscr{C}}(Y)$ in $\mathrm{Op}(\mathscr{V})$, there is a bijection between the morphisms $f \colon X \to Y$ in $\mathrm{Alg}_{\mathscr{C}}(\mathcal{O})$ and the morphisms $f \colon X \to Y$ in $\mathscr{C}$ such that there exists a morphism $\phi_f \colon \mathcal{O} \to \mathrm{End}_{\mathscr{C}^{\mathbf{1}}}(f)$ making the following diagram commutative*

In this corollary, the morphism ϕ_f is unique provided it exists since, by definition of end, we have pull-back diagrams as follows, $n \geq 0$,
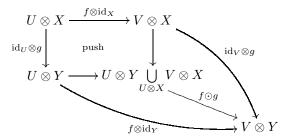

$$(1.9) \quad \mathrm{End}_{\mathscr{C}}(X)(n) = \mathrm{Hom}_{\mathscr{C}}(X^{\otimes n}, X)$$

2. HOMOTOPY INVARIANCE OF ENDOMORPHISM OPERADS

We here establish the homotopy invariance properties in $\mathrm{Op}(\mathscr{V})$ of endomorphism operads of objects in $\mathscr{C}$. This is a standard way to transfer algebra structures along weak equivalences in $\mathscr{C}$, compare [BM03, Theorem 3.5], and it will have further applications in Section 4.

Recall that an operad morphism $f \colon \mathcal{O} \to \mathcal{P}$ is a weak equivalence (resp. fibration) in $\mathrm{Op}(\mathscr{V})$ if all morphisms $f(n) \colon \mathcal{O}(n) \to \mathcal{P}(n)$ are weak equivalences (resp. fibrations) in $\mathscr{V}$, $n \geq 0$. Moreover, for any operad $\mathcal{O}$ in $\mathscr{V}$, an $\mathcal{O}$-algebra morphism $f \colon X \to Y$ is a weak equivalence (resp. fibration) in $\mathrm{Alg}_{\mathscr{C}}(\mathcal{O})$ if the underlying morphism in $\mathscr{C}$ is a weak equivalence (resp. fibration), see [Mur11a, Theorems 1.1 and 1.2].

Recall from [Mur11a, §4] that the category $\mathrm{Mor}(\mathscr{C})$ of morphisms in $\mathscr{C}$ carries a biclosed monoidal structure given by the $\odot$ product of morphisms $f \odot g$,

$$
\begin{array}{ccc}
U \otimes X & \xrightarrow{\ f \otimes \mathrm{id}_X\ } & V \otimes X \\
\scriptstyle{\mathrm{id}_U \otimes g}\downarrow & \quad\text{push} & \downarrow \scriptstyle{\mathrm{id}_V \otimes g} \\
U \otimes Y & \longrightarrow U \otimes Y \underset{U \otimes X}{\bigcup} V \otimes X & \\
& \searrow \scriptstyle{f \odot g} & \\
& \xrightarrow[f \otimes \mathrm{id}_Y]{} & V \otimes Y
\end{array}
$$

If U (resp. X) is the initial object 0 then $f \odot g = V \otimes g$ (resp. $f \otimes X$).

Definition 2.1. The *push-out product axiom* says that if f and g are cofibrations then so is $f \odot g$, and if either f or g is in fact a trivial cofibration then so is $f \odot g$.

The *monoid axiom* says that K'-cell complexes are weak equivalences, where K' is the following class of morphisms,

$$
\begin{aligned}
K' = \{ f_1 \odot \cdots \odot f_n \, ; & \, n \geq 1, S \subset \{1, \ldots, n\} \text{ is a non-empty subset} \\
& f_i \text{ is a trivial cofibration if } i \in S, \\
& f_i \colon 0 \to X_i \text{ for some object } X_i \text{ in } \mathscr{C} \text{ if } i \notin S \}.
\end{aligned}
$$

By the push-out product axiom, if Y is a cofibrant object in $\mathscr{C}$, the functor $z(-) \otimes Y$ is a left Quillen functor, therefore $\mathrm{Hom}_{\mathscr{C}}(Y, -)$ is a right Quillen functor, so it preserves fibrations, trivial fibrations, and fibrant objects, as well as weak equivalences between fibrant objects by Ken Brown's lemma. We now prove a similar property for the $\mathrm{Hom}_{\mathscr{C}}$ functor in the first variable.

Lemma 2.2. *If $f \colon X \to Y$ is a cofibration (resp. trivial cofibration) and Z is a fibrant object in $\mathscr{C}$, then the induced morphism*

$$
\mathrm{Hom}_{\mathscr{C}}(f, Z) \colon \mathrm{Hom}_{\mathscr{C}}(Y, Z) \longrightarrow \mathrm{Hom}_{\mathscr{C}}(X, Z)
$$

is a fibration (resp. trivial fibration).

Proof. Consider a commutative square of solid arrows in $\mathscr{V}$

$$
(2.3) \qquad
\begin{array}{ccc}
U & \xrightarrow{\ h\ } & \mathrm{Hom}_{\mathscr{C}}(Y, Z) \\
\scriptstyle{g}\downarrow & \overset{l}{\nearrow} & \downarrow \scriptstyle{\mathrm{Hom}_{\mathscr{C}}(f, Z)} \\
V & \xrightarrow[\ k\ ]{} & \mathrm{Hom}_{\mathscr{C}}(X, Z)
\end{array}
$$

where g is a trivial cofibration (resp. cofibration). We must construct a diagonal morphism l such that the two triangles above commute.

The solid diagram (2.3) is the same as a commutative square in $\mathscr{C}$

$$
\begin{array}{ccc}
z(U) \otimes X & \xrightarrow{\ z(U) \otimes f\ } & z(U) \otimes Y \\
\scriptstyle{z(g) \otimes X}\downarrow & & \downarrow \scriptstyle{\bar{h}} \\
z(V) \otimes X & \xrightarrow[\ \bar{k}\ ]{} & Z
\end{array}
$$

where $\bar{h}$ and $\bar{k}$ are the adjoints of h and k, respectively. We will consider the induced morphism from the push-out of the left upper corner to Z,

$$
\begin{array}{ccc}
z(U) \otimes X & \xrightarrow{z(U) \otimes f} & z(U) \otimes Y \\
{\scriptstyle z(g) \otimes X} \downarrow & {\scriptstyle \text{push}} & \downarrow {\scriptstyle \bar{g}} \\
z(V) \otimes X & \xrightarrow{\bar{f}} & W
\end{array}
$$

Since z is a left Quillen functor, $z(g)$ is a trivial cofibration (resp. cofibration), hence the morphism $z(g) \odot f \colon W \to z(V) \otimes Y$ is a trivial cofibration by the push-out product axiom. Since Z is fibrant, there exists a morphism $\bar{l} \colon z(V) \otimes Y \to Z$ fitting into a commutative triangle

$$
\begin{array}{ccc}
& W & \\
{\scriptstyle z(g) \odot f} \downarrow {\scriptstyle \sim} & \xrightarrow{r} & Z \\
z(V) \otimes Y & \xrightarrow{\bar{l}} &
\end{array}
$$

We can take $l \colon V \to \operatorname{Hom}_{\mathscr{C}}(Y, Z)$ to be the adjoint of $\bar{l}$. $\qquad \square$

The following result is a consequence of the previous lemma and Ken Brown's lemma.

Corollary 2.4. *For any fibrant object Z in $\mathscr{C}$, the functor $\operatorname{Hom}_{\mathscr{C}}(-, Z) \colon \mathscr{C}^{\mathrm{op}} \to \mathscr{V}$ takes weak equivalences between cofibrant objects in $\mathscr{C}$ to weak equivalences in $\mathscr{V}$.*

An immediate consequence of the following proposition is that the endomorphism operad of a fibrant-cofibrant object in $\mathscr{C}$ is an invariant of its weak homotopy type.

Proposition 2.5. *Let $f \colon X \to Y$ be a morphism in $\mathscr{C}$. Consider the induced morphisms between endomorphism operads in $\operatorname{Op}(\mathscr{V})$,*

$$
\operatorname{End}_{\mathscr{C}}(X) \xleftarrow{d_1} \operatorname{End}_{\mathscr{C}^1}(f) \xrightarrow{d_0} \operatorname{End}_{\mathscr{C}}(Y).
$$

(1) *If f is a (trivial) fibration and X is cofibrant then so is d_0.*

(2) *If f is a trivial fibration between fibrant-cofibrant objects then d_1 is a weak equivalence.*

(3) *If f is a (trivial) cofibration between cofibrant objects and Y is fibrant then d_1 is a (trivial) fibration.*

(4) *If f is a trivial cofibration between fibrant-cofibrant objects then d_0 is a weak equivalence.*

Proof. Consider the pull-back diagram (1.9). By the push-out product axiom, the tensor powers $X^{\otimes n}$ are cofibrant in the four cases. Hence, under the assumptions of (1) and (2), $\operatorname{Hom}_{\mathscr{C}}(X^{\otimes n}, f)$ is a (trivial) fibration. Now (1) follows from the fact that (trivial) fibrations are closed under pull-backs.

Under the hypotheses of (2), the push-out product axiom and Ken Brown's lemma show that the tensor powers $f^{\otimes n}$ are weak equivalences between cofibrant objects hence $\operatorname{Hom}_{\mathscr{C}}(f^{\otimes n}, Y)$ is a weak equivalence between fibrant objects by

Lemma 2.2 and Corollary 2.4. Therefore (2) follows, since the pull-back of a weak equivalence between fibrant objects along a fibration is a weak equivalence.

Under the assumptions of (3) and (4), the tensor powers $f^{\otimes n}$ are (trivial) cofibrations between cofibrant objects by the push-out product axiom. Hence, $\mathrm{Hom}_{\mathscr{C}}(f^{\otimes n}, Y)$ is a (trivial) fibration between fibrant objects by Lemma 2.2. Notice that (3) follows from the same reason as (1). Moreover, by Ken Brown's lemma $\mathrm{Hom}_{\mathscr{C}}(X^{\otimes n}, f)$ is a weak equivalence between fibrant objects, hence (4) follows from the same reason as (2). $\qquad\square$

The following corollary of Proposition 2.5 will be very useful in our study of moduli spaces of algebras.

Corollary 2.6. *Let $f_i \colon X_i \to X_{i+1}$ be trivial fibrations between fibrant-cofibrant objects in $\mathscr{C}$, $0 \leq i \leq n$. The induced morphisms between endomorphism operads in $\mathrm{Op}(\mathscr{V})$,*

$$\mathrm{End}_{\mathscr{C}^{\mathbf{n}}}(X_0 \to \cdots \to X_n) \xleftarrow{d_{n+1}} \mathrm{End}_{\mathscr{C}^{\mathbf{n+1}}}(X_0 \to \cdots \to X_n \to X_{n+1}) \xrightarrow{d_0^{n+1}} \mathrm{End}_{\mathscr{C}}(X_{n+1})$$

are a weak equivalence and a trivial fibration, respectively.

Proof. By induction on n. The case $n = 0$ follows directly from Proposition 2.5. Assume $n > 0$. Consider the following pull back diagram,

$$
\begin{array}{ccc}
\mathrm{End}_{\mathscr{C}^{\mathbf{n}}}(X_0 \to \cdots \to X_n)(m) & \xleftarrow{\quad d_{n+1} \quad} & \mathrm{End}_{\mathscr{C}^{\mathbf{n+1}}}(X_0 \to \cdots \to X_n \to X_{n+1})(m) \\
\downarrow {\scriptstyle d_0^n} & & \\
\mathrm{End}_{\mathscr{C}}(X_n)(m) & & \\
\| & \quad \text{pull} \quad & \downarrow {\scriptstyle d_0^{n+1}} \\
\mathrm{Hom}_{\mathscr{C}}(X_n^{\otimes m}, X_n) & & \\
\downarrow {\scriptstyle \mathrm{Hom}_{\mathscr{C}}(X_n^{\otimes m}, f_n)} & & \\
\mathrm{Hom}_{\mathscr{C}}(X_n^{\otimes m}, X_{n+1}) & \xleftarrow{\mathrm{Hom}_{\mathscr{C}}(f_n^{\otimes m}, X_{n+1})} & \mathrm{Hom}_{\mathscr{C}}(X_{n+1}^{\otimes m}, X_{n+1}) = \mathrm{End}_{\mathscr{C}}(X_{n+1})
\end{array}
$$

The morphism d_0^n is a trivial fibration by induction hypothesis. We showed within the proof of Proposition 2.5 that, under the hypotheses of this corollary, the morphisms $\mathrm{Hom}_{\mathscr{C}}(X_n^{\otimes m}, f_n)$ and $\mathrm{Hom}_{\mathscr{C}}(f_n^{\otimes m}, X_{n+1})$ are a trivial fibration and a weak equivalence between fibrant objects, respectively. Hence, this result follows from the facts that trivial fibrations are closed under composition and pull-backs and weak equivalences between fibrant objects are closed under pull-backs along fibrations. $\qquad\square$

3. Quillen equivalences between categories of algebras

A morphism of operads $\phi \colon \mathcal{O} \to \mathcal{P}$ in $\mathrm{Op}(\mathscr{V})$ induces a Quillen pair of change of operad functors,

$$(3.1) \qquad\qquad \mathrm{Alg}_{\mathscr{C}}(\mathcal{O}) \underset{\phi^*}{\overset{\phi_*}{\rightleftarrows}} \mathrm{Alg}_{\mathscr{C}}(\mathcal{P}).$$

The functor ϕ^* restricts the action of $\mathcal{P}$ to $\mathcal{O}$ along ϕ and is the identity on underlying objects in $\mathscr{C}$, hence it preserves fibrations and weak equivalences. The functor ϕ_* is left adjoint to ϕ^*.

We are interested in conditions on ϕ under which this pair is a Quillen equivalence. A reasonable condition is that ϕ be a weak equivalence in $\mathrm{Op}(\mathscr{V})$, but often this is not enough. In [Mur11a] we gave some sufficient conditions, but our assumptions were not general enough for applications in homotopical algebraic geometry, where monoidal model categories with non-cofibrant unit arise.

The issue is that, if $\mathcal{O}$ is a cofibrant operad in $\mathrm{Op}(\mathscr{V})$ then $\mathcal{O}(n)$ is cofibrant if $n \neq 1$, and for $n = 1$ the unit $u\colon \mathbb{I} \to \mathcal{O}(1)$ is a cofibration, see Corollary 3.13 below. In particular, if $\mathbb{I}$ is cofibrant then so is $\mathcal{O}(1)$, but otherwise $\mathcal{O}(1)$ need not be cofibrant, which is a hypothesis in [Mur11a, Theorem 1.3].

In homotopical algebraic geometry one needs the category of commutative monoids in $\mathscr{V}$ to be a model category with fibrations and weak equivalences defined as in $\mathscr{V}$. If $\mathscr{V}$ is the category of chain complexes over a commutative ring $\Bbbk$ then the most common sufficient condition in order to guarantee this is that $\Bbbk$ be a $\mathbb{Q}$-algebra. This is a very strong condition that we do not want to assume. The standard way to overcome this difficulty is to replace $\mathscr{V}$ with the Quillen equivalent category of modules over the Eilenberg–MacLane symmetric ring spectrum associated to $\Bbbk$ with Shipley's positive stable model structure [Shi04]. The monoidal unit is unfortunately not cofibrant in this category. There are actually obstructions for the existence of a model of $\mathscr{V}$ with all these good properties, compare [Lew91].

We will see in the main result of this section that weak equivalences between operads satisfying the following property induce a Quillen equivalence (3.1). This generalizes [Mur11a, Theorem 1.3].

Definition 3.2. A *weakly cofibrant operad* is an operad $\mathcal{O}$ in $\mathscr{V}$ such that $\mathcal{O}(n)$ is cofibrant if $n \neq 1$ and the unit $u\colon \mathbb{I} \to \mathcal{O}(1)$ is a cofibration.

In order to obtain the desired generalization we need the following variant of Ken Brown's lemma.

Lemma 3.3. *Let $F\colon \mathscr{M} \to \mathscr{W}$ be a functor from a model category $\mathscr{M}$ to a category $\mathscr{W}$ with a class of weak equivalences satisfying the 2-out-of-3 axiom. Assume that F takes trivial cofibrations to weak equivalences. If $g\colon Y \to Z$ is a weak equivalence in $\mathscr{M}$ fitting into a commutative triangle*

$$
\begin{array}{ccc}
 & & Y \\
 & \nearrow & \\
X & & \Big\downarrow {\scriptstyle g} \\
 & \searrow & \\
 & & Z
\end{array}
$$

then the morphism $F(g)$ is a weak equivalence in $\mathscr{W}$.

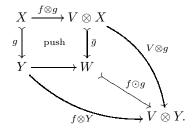*Proof.* Apply Ken Brown's lemma to the composite

$$ X \downarrow \mathscr{M} \longrightarrow \mathscr{M} \xrightarrow{\ F\ } \mathscr{W}, $$

where $X \downarrow \mathscr{M}$ is the category of objects under X and the first functor is the functor which forgets the morphism from X. The model structure on $X \downarrow \mathscr{M}$ is reflected by this forgetful functor. $\qquad\square$

The following lemma is the key observation.

Lemma 3.4. *If $g\colon X \rightarrowtail Y$ is a (trivial) cofibration and V is an object in $\mathscr{C}$ which admits a cofibration from the monoidal unit $f\colon \mathbb{I} \rightarrowtail V$ then $V \otimes g\colon V \otimes X \rightarrowtail V \otimes Y$ and $g \otimes V\colon X \otimes V \rightarrowtail Y \otimes V$ are also (trivial) cofibrations.*

Proof. Consider the diagramatic construction of $f \odot g$ where, for simplicity, we identify $\mathbb{I} \otimes -$ with the identity functor,

$$
\begin{array}{ccc}
X & \xrightarrow{\ f\otimes g\ } & V \otimes X \\
\end{array}
$$

Here $\bar{g}$ is a (trivial) cofibration because it is the push-out of a (trivial) cofibration along a map, and $f \odot g$ is a (trivial) cofibration by the push-out product axiom, therefore $V \otimes g = (f \odot g)\bar{g}$ is a (trivial) cofibration. One similarly checks that $g \otimes V$ is also a (trivial) cofibration. $\qquad\square$

Given morphisms $f_i\colon U_i \to V_i$ in $\mathscr{C}$, $1 \le i \le n$, the source of $f_1 \odot \cdots \odot f_n$, denoted $s(f_1 \odot \cdots \odot f_n)$, is the colimit of a diagram indexed by the category with objects (i,j), $n \ge i \ge j \ge 1$, whose only non-identity morphisms are $(i,i) \leftarrow (i,j) \to (j,j)$, $n \ge i > j \ge 1$. Notice that two non-identity morphisms can never be composed. The objects of this diagram are

$$(i,i) \mapsto V_1 \otimes \cdots \otimes V_{i-1} \otimes U_i \otimes V_{i+1} \otimes \cdots \otimes V_n,$$

$$(i,j) \mapsto V_1 \otimes \cdots \otimes V_{j-1} \otimes U_j \otimes V_{j+1} \otimes \cdots \otimes V_{i-1} \otimes U_i \otimes V_{i+1} \otimes \cdots \otimes V_n, \quad i > j,$$

and the morphisms induced by $(i,i) \leftarrow (i,j) \to (j,j)$ are $\mathrm{id}^{\otimes(j-1)} \otimes f_j \otimes \mathrm{id}^{\otimes(n-j)}$ and $\mathrm{id}^{\otimes(i-1)} \otimes f_i \otimes \mathrm{id}^{\otimes(n-i)}$, respectively. The colimit of this diagram can be inductively constructed by means of $\binom{n}{2}$ push-outs.

Lemma 3.5. *Suppose that $\mathscr{C}$ is left proper. Consider n commutative squares in $\mathrm{Mor}(\mathscr{C})$ where the rows are cofibrations and the columns are weak equivalences satisfying the assumption in Lemma 3.3, $1 \le i \le n$,*

$$
\begin{array}{ccc}
U_i & \xrightarrow{\ f_i\ } & V_i \\
{\scriptstyle g_i}\downarrow{\scriptstyle \sim} & & {\scriptstyle \sim}\downarrow{\scriptstyle g_i'} \\
X_i & \xrightarrow[\ f_i'\ ]{} & Y_i
\end{array}
$$

Then in the following diagram the rows are also cofibrations and the columns are also weak equivalences,

$$
\begin{array}{ccc}
s(f_1 \odot \cdots \odot f_n) & \xrightarrow{\ f_1 \odot \cdots \odot f_n\ } & V_1 \otimes \cdots \otimes V_n \\
{\scriptstyle \sim}\downarrow & & {\scriptstyle \sim}\downarrow{\scriptstyle g_1' \otimes \cdots \otimes g_n'} \\
s(f_1' \odot \cdots \odot f_n') & \xrightarrow[\ f_1' \odot \cdots \odot f_n'\ ]{} & Y_1 \otimes \cdots \otimes Y_n
\end{array}
$$

Proof. The rows are cofibrations by the push-out product axiom. The right vertical arrow decomposes as

$$g'_1 \otimes \cdots \otimes g'_n = (g'_1 \otimes \mathrm{id}^{\otimes(n-1)}) \cdots (\mathrm{id}^{\otimes(i-1)} \otimes g'_i \otimes \mathrm{id}^{\otimes(n-i)}) \cdots (\mathrm{id}^{\otimes(n-1)} \otimes g'_n).$$

Lemma 3.3 and the monoid axiom show the ith factor is a weak equivalence, so $g'_1 \otimes \cdots \otimes g'_n$ is a weak equivalence too.

Since $\mathscr{C}$ is left proper, the same argument shows that the left vertical arrow can be inductively constructed by taking push-out in $\binom{n}{2}$ diagrams of the form

Therefore it is a weak equivalence by the gluing property for weak equivalences in left proper model categories. $\qquad\square$

Proposition 3.6. *Let $\mathcal{O}$ be a weakly cofibrant operad in $\mathscr{V}$. Consider a push-out diagram in $\mathrm{Alg}_{\mathscr{C}}(\mathcal{O})$ as follows.*

$$
\begin{array}{ccc}
\mathcal{F}_{\mathcal{O}}(Y) & \xrightarrow{\mathcal{F}_{\mathcal{O}}(f)} & \mathcal{F}_{\mathcal{O}}(Z) \\
{\scriptstyle g}\downarrow & {\scriptstyle push} & \downarrow{\scriptstyle g'} \\
A & \xrightarrow{f'} & B
\end{array}
$$

Suppose that A is cofibrant in $\mathscr{C}$ and f is a cofibration. Then the morphism $f'\colon A \to B$ is a cofibration in $\mathscr{C}$, in particular B is cofibrant in $\mathscr{C}$.

Proof. Since z is a left Quillen functor, the objects $z(\mathcal{O}(n))$ are cofibrant in $\mathscr{C}$ for $n \neq 1$, and $z(\mathcal{O}(1))$ admits a cofibration from the tensor unit. Therefore, by the push-out product axiom and Lemma 3.4, the morphism [Mur11a, (15)] is a cofibration in $\mathscr{C}$. Furthermore, by [Mur11a, Theorem 8.3] the morphism $f'\colon A \to B$ is a transfinite composition of cofibrations in $\mathscr{C}$, hence a cofibration in $\mathscr{C}$ itself. $\quad\square$

Lemma 3.7. *Suppose that $\mathcal{O}$ is a weakly cofibrant operad in $\mathscr{V}$. Then any cofibrant $\mathcal{O}$-algebra in $\mathscr{C}$ is also a cofibrant object in $\mathscr{C}$.*

The proof of this lemma is the same as the proof of [Mur11a, Lemma 9.4], but applying Proposition 3.6 instead of [Mur11a, Proposition 9.2 (2)].

Corollary 3.8. *Let $\mathcal{O}$ be a weakly cofibrant operad in $\mathscr{V}$. Then, the forgetful functor $\mathrm{Alg}_{\mathscr{C}}(\mathcal{O}) \to \mathscr{C}$ preserves cofibrations with cofibrant source.*

Compare [Mur11a, Corollary 9.5].

Lemma 3.9. *Suppose $\mathscr{C}$ is left proper, $\phi\colon \mathcal{O} \xrightarrow{\sim} \mathcal{P}$ is a weak equivalence between weakly cofibrant operads in $\mathscr{V}$, and we have a push-out diagram in $\mathrm{Alg}_{\mathscr{C}}(\mathcal{O})$,*

$$
\begin{array}{ccc}
\mathcal{F}_{\mathcal{O}}(Y) & \rightarrowtail^{\mathcal{F}_{\mathcal{O}}(f)} & \mathcal{F}_{\mathcal{O}}(Z) \\
{\scriptstyle g}\downarrow & {\scriptstyle push} & \downarrow{\scriptstyle g'} \\
A & \rightarrowtail_{f'} & B
\end{array}
$$

where f is a cofibration in $\mathscr{C}$ and A is a cofibrant $\mathcal{O}$-algebra. If the unit η of the adjunction (3.1) evaluated at A is a weak equivalence $\eta_A \colon A \xrightarrow{\sim} \phi^* \phi_* A$, then it is also a weak equivalence when evaluated at B, $\eta_B \colon B \xrightarrow{\sim} \phi^* \phi_* B$.

The proof of this lemma is the same as the proof of [Mur11a, Lemma 9.6] with the following modifications: A and $\phi_* A$ are cofibrant in $\mathscr{C}$ by Lemma 3.7, the push-out product axiom must be applied together with Lemma 3.4, and [Mur11a, Lemma 4.4] must be replaced with Lemma 3.5.

Remark 3.10. In the proof of [Mur11a, Lemma 9.6] we invoke [Mur11a, Lemma 4.4], which does not apply if Y is not cofibrant, but Lemma 3.5 above does always apply.

The following theorem is the main result of this section.

Theorem 3.11. *If $\mathscr{C}$ is left proper then (3.1) is a Quillen equivalence. In particular the derived adjoint pair is an equivalence between the homotopy categories of algebras,*

$$\operatorname{Ho} \operatorname{Alg}_{\mathscr{C}}(\mathcal{O}) \xhookleftarrow[\phi^*]{\mathbb{L}\phi_*} \operatorname{Ho} \operatorname{Alg}_{\mathscr{C}}(\mathcal{P}).$$

The proof of this result is the same as the proof of [Mur11a, Theorem 1.3] applying Proposition 3.6 instead of [Mur11a, Proposition 9.2 (2)].

The following two results show that there are enough weakly cofibrant operads in $\operatorname{Op}(\mathscr{V})$.

Lemma 3.12. *If $\phi \colon \mathcal{P} \rightarrowtail \mathcal{O}$ is a cofibration in $\operatorname{Op}(\mathscr{V})$ with weakly cofibrant source, then $\phi(n) \colon \mathcal{P}(n) \rightarrowtail \mathcal{O}(n)$ is a cofibration in $\mathscr{V}$ for all $n \geq 0$. In particular $\mathcal{O}$ is also weakly cofibrant.*

Proof. Denote I a set of generating cofibrations with presentable sources in $\mathscr{V}$. Recall from [Mur11a] that the forgetful functors $\mathscr{V}^{\mathbb{N}} \to \mathscr{V} \colon \{U_n\}_{n \geq 0} \mapsto U_i$ have left adjoints $s_i \colon \mathscr{V} \to \mathscr{V}^{\mathbb{N}}$, $i \geq 0$, and $I_{\mathbb{N}} = \cup_{i \geq 0} s_i(I)$ is a set of generating cofibrations in $\mathscr{V}^{\mathbb{N}}$. Moreover, $\mathcal{F}(I_{\mathbb{N}})$ is a set of generating cofibrations in $\operatorname{Op}(\mathscr{V})$. We can therefore suppose that ϕ is a relative $\mathcal{F}(I_{\mathbb{N}})$-cell complex. In this case, the description of push-outs along free morphisms in $\operatorname{Op}(\mathscr{V})$ [Mur11a, Theorem 5.4], the push-out product axiom, and Lemma 3.4, show that $\phi(n)$ is a relative $c\mathscr{V}$-cell complex, where $c\mathscr{V} \subset \mathscr{V}$ is the subcategory of cofibrations, and hence a cofibration in $\mathscr{V}$. $\square$

Corollary 3.13. *Cofibrant operads in $\mathscr{V}$ are weakly cofibrant.*

This follows from the fact that the initial operad $\mathcal{P}$ is given by $\mathcal{P}(n) = 0$ the initial object for $n \neq 1$, and the unit $u \colon \mathbb{I}_{\mathscr{V}} \to \mathcal{P}(1)$ is the identity, so it is weakly cofibrant.

4. Mapping spaces

If $\mathcal{O}^{\bullet}$ is a cosimplicial operad then the change of base operad functors like ϕ^* in (3.1) give rise to a simplicial category $\operatorname{Alg}_{\mathscr{C}}(\mathcal{O}^{\bullet})$. Recall from [DK80, §4.3] the notion of cosimplicial resolution. If $\mathscr{W}$ is a category with weak equivalences, we denote $w\mathscr{W}$ the subcategory of weak equivalences. From now on, we assume further that $\mathscr{C}$ is left proper.

Lemma 4.1. *If $\mathcal{O}^\bullet$ is a cosimplicial resolution of a weakly cofibrant operad $\mathcal{O}$ in $\mathcal{V}$, then there is a weak equivalence*

$$|w\operatorname{Alg}_{\mathscr{C}}(\mathcal{O})| \xrightarrow{\sim} |w\operatorname{Alg}_{\mathscr{C}}(\mathcal{O}^\bullet)|.$$

Proof. By Theorem 3.11 all faces and degeneracies in $\operatorname{Alg}_{\mathscr{C}}(\mathcal{O}^\bullet)$ are right adjoints of a Quillen equivalence, therefore all faces and degeneracies in $w\operatorname{Alg}_{\mathscr{C}}(\mathcal{O}^\bullet)$ induce weak equivalences on nerves. This implies that the iterated degeneracies induce a weak equivalence

$$|w\operatorname{Alg}_{\mathscr{C}}(\mathcal{O}^0)| \xrightarrow{\sim} |w\operatorname{Alg}_{\mathscr{C}}(\mathcal{O}^\bullet)|.$$

For the same reason the weak equivalence $\mathcal{O}^0 \xrightarrow{\sim} \mathcal{O}$ induces a weak equivalence on nerves,

$$|w\operatorname{Alg}_{\mathscr{C}}(\mathcal{O})| \xrightarrow{\sim} |w\operatorname{Alg}_{\mathscr{C}}(\mathcal{O}^0)|.$$

$\square$

Denote $fw\mathscr{C}_{fc}$ the category of fibrant-cofibrant objects in $\mathscr{C}$ and trivial fibrations between them. Under the assumptions of the previous lemma, define the simplicial category $\mathscr{D}_\bullet$ as the following pull-back,

(4.2)
$$
\begin{array}{ccc}
\mathscr{D}_\bullet & \longrightarrow & w\operatorname{Alg}_{\mathscr{C}}(\mathcal{O}^\bullet) \\
\rho'_\bullet \downarrow & \text{pull} & \downarrow \rho_\bullet \\
fw\mathscr{C}_{fc} & \xrightarrow[\text{inclusion}]{} & \mathscr{C}
\end{array}
$$

Here we regard the two categories in the bottom as constant simplicial categories, and $\rho_\bullet$ is the forgetful simplicial functor. Notice that $\mathscr{D}_n$ is the category of trivial fibrations between $\mathcal{O}^n$-algebras whose underlying objects in $\mathscr{C}$ are fibrant and cofibrant.

Lemma 4.3. *The inclusion induces a weak equivalence,*

$$|fw\mathscr{C}_{fc}| \xrightarrow{\sim} |w\mathscr{C}|.$$

Moreover, given a cosimplicial resolution $\mathcal{O}^\bullet$ in $\operatorname{Op}(\mathcal{V})$, the upper row in (4.2) *induces a weak equivalence,*

$$|\mathscr{D}_\bullet| \xrightarrow{\sim} |w\operatorname{Alg}_{\mathscr{C}}(\mathcal{O}^\bullet)|.$$

Proof. The first part of the statement follows from [Rez96, Lemmas 4.2.4 and 4.2.5]. Moreover, by the same results the inclusion $\mathscr{D}_n \subset w\operatorname{Alg}_{\mathscr{C}}(\mathcal{O}^n)$ induces a weak equivalence

$$|\mathscr{D}_n| \xrightarrow{\sim} |w\operatorname{Alg}_{\mathscr{C}}(\mathcal{O}^n)|, \quad n \geq 0,$$

hence the second map in the statement is also a weak equivalence. $\square$

Recall that the *category of simplicies* ΔK of a simplicial set K is the category whose objects are pairs $(\mathbf{n}, x)$ with $n \geq 0$ and $x \in K_n$. A morphism $\sigma\colon (\mathbf{n}, x) \to (\mathbf{m}, y)$ in ΔK is a morphism $\sigma\colon \mathbf{n} \to \mathbf{m}$ in Δ such that the induced map $\sigma^*\colon K_m \to K_n$ takes y to x, $\sigma^*(y) = x$. This category comes equipped with a natural projection functor $p_K\colon \Delta K \to \Delta\colon (\mathbf{n}, x) \mapsto \mathbf{n}$. This construction defines a functor from the category of simplicial sets to the category of categories over Δ,

$$
\begin{aligned}
\mathbf{Set}^{\Delta^{\mathrm{op}}} &\longrightarrow \mathbf{Cat} \downarrow \Delta, \\
K &\mapsto p_K.
\end{aligned}
$$

Lemma 4.4. *If Y is a fibrant-cofibrant object in $\mathscr{C}$ and $\mathcal{O}^\bullet$ is a cosimplicial resolution in $\mathrm{Op}(\mathscr{V})$, then $|\rho'_\bullet \downarrow Y|$ is weakly equivalent to $\mathrm{Op}(\mathscr{V})(\mathcal{O}^\bullet, \mathrm{End}_{\mathscr{C}}(Y))$.*

Proof. We will use an alternative construction of the bisimplicial set $|\rho'_\bullet \downarrow Y|$ in terms of endomorphism operads. Clearly,

$$(4.5) \qquad |\rho'_t \downarrow Y|_s = \coprod_{X_0 \to \cdots \to X_s \to Y \text{ in } fw\mathscr{C}_{fc}} \mathrm{Op}(\mathscr{V})(\mathcal{O}^t, \mathrm{End}_{\mathscr{C}^s}(X_0 \to \cdots \to X_s)).$$

Notice that the set indexing this coproduct is $|(fw\mathscr{C}_{fc}) \downarrow Y|_s$. In order to describe $|\rho'_\bullet \downarrow Y|$ in terms of the right hand side of (4.5), we consider the functor

$$E \colon (\Delta|\mathscr{C}|)^{\mathrm{op}} \longrightarrow \mathrm{Op}(\mathscr{V}),$$
$$(\mathbf{n}, X_0 \to \cdots \to X_n) \mapsto \mathrm{End}_{\mathscr{C}^n}(X_0 \to \cdots \to X_n).$$

Notice that $X_0 \to \cdots \to X_n$ is a functor $X \colon \mathbf{n} \to \mathscr{C}$. Given a morphism $\sigma \colon (\mathbf{n}, X) \to (\mathbf{m}, X')$ in the category of simplices, with the notation introduced in Section 1, the induced morphism $E(\sigma)$ is $\sigma^* \colon \mathrm{End}_{\mathscr{C}^m}(X') \to \mathrm{End}_{\mathscr{C}^n}(X)$. We also consider the functor

$$F_Y \colon (fw\mathscr{C}_{fc}) \downarrow Y \longrightarrow \mathscr{C},$$
$$(X \to Y) \mapsto X,$$

and the composite functor

$$(\Delta|(fw\mathscr{C}_{fc}) \downarrow Y|)^{\mathrm{op}} \xrightarrow{(\Delta|F_Y|)^{\mathrm{op}}} (\Delta|\mathscr{C}|)^{\mathrm{op}} \xrightarrow{E} \mathrm{Op}(\mathscr{V}) \xrightarrow{\mathrm{Op}(\mathscr{V})(\mathcal{O}^\bullet, -)} \mathbf{Set}^{\Delta^{\mathrm{op}}}.$$

Taking the left Kan extension [Mac98, X.3] of this functor along the opposite of the canonical proyection from the category of simplices to Δ, we obtain a bisimplicial set

$$\mathrm{Lan}_{p^{\mathrm{op}}_{|(fw\mathscr{C}_{fc})\downarrow Y|}} \mathrm{Op}(\mathscr{V})(\mathcal{O}^\bullet, E(\Delta|F_Y|)^{\mathrm{op}}).$$

One can easily check that the (s, t) set of this bisimplicial set is the right hand side of (4.5). Moreover, this defines an isomorphism between this bisimplicial set and $|\rho'_\bullet \downarrow Y|$.

We need two more functors

$$L_Y, C_Y \colon \Delta|(fw\mathscr{C}_{fc}) \downarrow Y| \longrightarrow \Delta|\mathscr{C}|,$$
$$L_Y(\mathbf{n}, X_0 \to \cdots \to X_n \to Y) = (\mathbf{n+1}, X_0 \to \cdots \to X_n \to Y),$$
$$C_Y(\mathbf{n}, X_0 \to \cdots \to X_n \to Y) = (\mathbf{0}, Y),$$

and two natural transformations,

$$\delta_Y \colon \Delta|F_Y| \Longrightarrow L_Y, \qquad \delta_Y(\mathbf{n}, X_0 \to \cdots \to X_n \to Y) = d^{n+1},$$
$$\varsigma_Y \colon C_Y \Longrightarrow L_Y, \qquad \varsigma_Y(\mathbf{n}, X_0 \to \cdots \to X_n \to Y) = (d^0)^{n+1}.$$

For simplicity, denote $K = |(fw\mathscr{C}_{fc}) \downarrow Y|$. We claim that the following morphisms of bisimplicial sets are weak equivalences,

$$\mathrm{Lan}_{p_K^{\mathrm{op}}} \mathrm{Op}(\mathscr{V})(\mathcal{O}^\bullet, E(\Delta|F_Y|)^{\mathrm{op}})$$

$$\Big\uparrow {\scriptstyle \mathrm{Lan}_{p_K^{\mathrm{op}}} \mathrm{Op}(\mathscr{V})(\mathcal{O}^\bullet, E(\delta_Y))}$$

$$\mathrm{Lan}_{p_K^{\mathrm{op}}} \mathrm{Op}(\mathscr{V})(\mathcal{O}^\bullet, EL_Y^{\mathrm{op}})$$

$$\Big\downarrow {\scriptstyle \mathrm{Lan}_{p_K^{\mathrm{op}}} \mathrm{Op}(\mathscr{V})(\mathcal{O}^\bullet, E(\varsigma_Y))}$$

$$\mathrm{Lan}_{p_K^{\mathrm{op}}} \mathrm{Op}(\mathscr{V})(\mathcal{O}^\bullet, EC_Y^{\mathrm{op}})$$

It is enough to notice that, at the $(n, \bullet)$ simplicial set, we have the coproduct indexed by $|(fw\mathscr{C}_{fc}) \downarrow Y|_n$ of the morphisms of simplicial sets obtained by applying $\mathrm{Op}(\mathscr{V})(\mathcal{O}^\bullet, -)$ to the weak equivalences of operads in Corollary 2.6, $n \geq 0$,

$$\coprod_{X_0 \to \cdots \to X_n \to Y \text{ in } fw\mathscr{C}_{fc}} \mathrm{Op}(\mathscr{V})(\mathcal{O}^\bullet, \mathtt{End}_{\mathscr{C}^\mathbf{n}}(X_0 \to \cdots \to X_n))$$

$$\Big\uparrow {\scriptstyle (d_{n+1})_*} \ \sim$$

$$\coprod_{X_0 \to \cdots \to X_n \to Y \text{ in } fw\mathscr{C}_{fc}} \mathrm{Op}(\mathscr{V})(\mathcal{O}^\bullet, \mathtt{End}_{\mathscr{C}^{\mathbf{n+1}}}(X_0 \to \cdots \to X_n \to Y))$$

$$\Big\downarrow {\scriptstyle (d_0^{n+1})_*} \ \sim$$

$$\coprod_{X_0 \to \cdots \to X_n \to Y \text{ in } fw\mathscr{C}_{fc}} \mathrm{Op}(\mathscr{V})(\mathcal{O}^\bullet, \mathtt{End}_{\mathscr{C}}(Y))$$

These arrows are weak equivalences by [DK80, Corollary 6.4].

In order to complete the definition of the weak equivalence claimed in the statement we notice that

$$\mathrm{Lan}_{p_K^{\mathrm{op}}} \mathrm{Op}(\mathscr{V})(\mathcal{O}^\bullet, EC_Y^{\mathrm{op}}) = |(fw\mathscr{C}_{fc}) \downarrow Y| \times \mathrm{Op}(\mathscr{V})(\mathcal{O}^\bullet, \mathtt{End}_{\mathscr{C}}(Y)).$$

The category $(fw\mathscr{C}_{fc}) \downarrow Y$ has a final object, the identity in Y, hence it has a contractible nerve and the projection onto the second factor of the product gives rise to a weak equivalence

$$\mathrm{Lan}_{p_K^{\mathrm{op}}} \mathrm{Op}(\mathscr{V})(\mathcal{O}^\bullet, E(\Delta|C_Y|)^{\mathrm{op}}) \xrightarrow{\sim} \mathrm{Op}(\mathscr{V})(\mathcal{O}^\bullet, \mathtt{End}_{\mathscr{C}}(Y)).$$

$$\square$$

We can now proceed with the main theorem of this section.

Theorem 4.6. *Let $\mathcal{O}$ be a weakly cofibrant operad in $\mathscr{V}$. The homotopy fiber of the morphism induced by the forgetful functor $\mathrm{Alg}_{\mathcal{O}}(\mathscr{C}) \to \mathscr{C}$,*

$$|w\,\mathrm{Alg}_{\mathscr{C}}(\mathcal{O})| \longrightarrow |w\mathscr{C}|,$$

at a fibrant-cofibrant object Y is $\mathrm{Map}_{\mathrm{Op}(\mathscr{V})}(\mathcal{O}, \mathtt{End}_{\mathscr{C}}(Y))$.

Proof. By Lemma 4.4, we can apply [Rez96, Lemma 4.2.2] to the simplicial functor $\rho'_\bullet: \mathscr{D}_\bullet \to fw\mathscr{C}_{fc}$. This, together with Lemmas 4.1 and 4.3, proves the statement.

$$\square$$

5. Geometric applications

In this section we place ourselves in a homotopical algebraic geometry (HAG) context [TV08, 1.3.2.13] with underlying symmetric monoidal model category $\mathcal{V}$. In particular $\mathcal{V}$ is combinatorial, proper and pointed, and the category $\mathrm{Comm}(\mathcal{V})$ of commutative algebras (i.e. monoids) A in $\mathcal{V}$ and their module categories $\mathrm{Mod}(A)$ inherit combinatorial proper model category structures with fibrations and weak equivalences defined by the underlying morphisms in $\mathcal{V}$. Actually $\mathrm{Mod}(A)$ inherits all the structure from $\mathcal{V}$, and is itself the underlying symmetric monoidal model category of a new induced HAG context. Hovey's unit axiom must hold, i.e. tensoring a cofibrant replacement of $\mathbb{I}$ with a cofibrant object yields a weak equivalence, see [Hov99, Definition 4.2.6 (2) and Lemma 4.2]. Notice that since $\mathcal{V}$ is combinatorial, it is λ-presentable for a certain regular cardinal λ.

The category $\mathrm{Aff}_{\mathcal{V}}$ of *affine stacks* is opposite to $\mathrm{Comm}(\mathcal{V})$. A *simplicial presheaf* is a (contravariant) functor from affine stacks to simplicial sets,

$$\mathrm{Aff}_{\mathcal{V}}^{\mathrm{op}} \longrightarrow \mathbf{Set}^{\Delta^{\mathrm{op}}},$$

i.e. a (covariant) functor from $\mathrm{Comm}(\mathcal{V})$. The category $\mathrm{SPr}(\mathrm{Aff}_{\mathcal{V}})$ of simplicial presheaves carries a model structure where weak equivalences (resp. fibrations) are pointwise weak equivalences (resp. fibrations) of simplicial sets.

The simplicial presheaf represented by a commutative algebra A is

$$\mathbb{R}\underline{\mathrm{Spec}}(A) = \mathrm{Map}_{\mathrm{Comm}(\mathcal{V})}(A, -).$$

This functor is defined by a cosimplicial resolution of A and a functorial fibrant resolution in $\mathrm{Comm}(\mathcal{V})$. In order to turn $\mathbb{R}\underline{\mathrm{Spec}}$ into a functor

$$\mathbb{R}\underline{\mathrm{Spec}} \colon \mathrm{Aff}_{\mathcal{V}} \longrightarrow \mathrm{SPr}(\mathrm{Aff}_{\mathcal{V}})$$

we must choose a functorial cofibrant resolution in $\mathrm{Comm}(\mathcal{V})$. The model category of prestacks $\mathrm{Aff}_{\mathcal{V}}^{\wedge}$ is the left Bousfield localization of $\mathrm{SPr}(\mathrm{Aff}_{\mathcal{V}})$ with respect to the image by $\mathbb{R}\underline{\mathrm{Spec}}$ of weak equivalences between affine stacks.

As part of our HAG context, we have a subcanonical model topology τ on $\mathrm{Aff}_{\mathcal{V}}$. The model category of stacks $\mathrm{Aff}_{\mathcal{V}}^{\sim,\tau}$ is a left Bousfield localization of $\mathrm{Aff}_{\mathcal{V}}^{\wedge}$ with respect to a certain set of hypercovers, see [TV05, Corollary 4.6.2]. We regard $\mathrm{Ho}\,\mathrm{Aff}_{\mathcal{V}}^{\sim,\tau}$ as a full subcategory of $\mathrm{Ho}\,\mathrm{Aff}_{\mathcal{V}}^{\wedge}$ through the derived right Quillen functor of this left Bousfield localization. We similarly regard $\mathrm{Ho}\,\mathrm{Aff}_{\mathcal{V}}^{\wedge}$ as a full subcategory of $\mathrm{Ho}\,\mathrm{SPr}(\mathrm{Aff}_{\mathcal{V}})$. Representable simplicial presheaves are stacks, called *representable* or *affine stacks*. The final stack $*$ is representable, $* = \mathbb{R}\underline{\mathrm{Spec}}(\mathbb{I})$.

Geometric stacks are defined from a certain class of morphisms $\mathbf{P}$ in $\overline{\mathrm{Aff}_{\mathcal{V}}}$, which is also part of the HAG context. Intuitively, quoting Toën and Vezzosi, geometric stacks are quotients of representable stacks by equivalence relations whose structural morphisms are in $\mathbf{P}$. The infinitesimal theory is controlled by full subcategories $\mathcal{V}_0 \subset \mathcal{V}$ and $\mathscr{A} \subset \mathrm{Comm}(\mathcal{V})$ playing the role of the aisle of a t-structure in a tensor triangulated category and rings in the heart of the t-structure, respectively.

Example 5.1 ([TV08, §2.3.2, §2.3.4, §2.4.1]). Some examples of HAG contexts we are interested in are:

(1) $\mathcal{V} = \mathcal{V}_0$ the category of complexes over a $\mathbb{Q}$-algebra $\Bbbk$ (we use homological grading, i.e. degree -1 differentials) where weak equivalences are quasi-isomorphisms and fibrations are levelwise surjective morphisms, $\mathscr{A} = \mathrm{Comm}(\mathcal{V})$ the whole category of differential graded commutative (DGC)

$\Bbbk$-algebras, $\mathbf{P}$ the formally perfect morphisms, i.e. morphisms of DGC $\Bbbk$-algebras $A \to B$ whose cotangent complex $\mathbb{L}_{B/A}$ is perfect, and τ the strongly étale topology, whose coverings are sets of morphisms $\{A \to A_i\}_{i \in I}$ of DGC $\Bbbk$-algebras such that, for all $i \in I$, $H_0(A) \to H_0(A_i)$ is an étale $\Bbbk$-algebra homomorphism, the morphism

$$H_*(A) \otimes_{H_0(A)} H_0(A_i) \longrightarrow H_*(A_i)$$

is an isomorphism of graded $H_0(A_i)$-algebras, and there exists a finite subset $J \subset I$ such that the left derived change of coefficient functors

$$(- \otimes_A^{\mathbb{L}} A_j)_{j \in J} \colon \operatorname{Ho} \operatorname{Mod}(A) \longrightarrow \prod_{j \in J} \operatorname{Ho} \operatorname{Mod}(A_j)$$

reflect isomorphisms.

(2) $\mathscr{V}$ and τ as above, $\mathscr{V}_0$ the category of complexes M with $H_n M = 0$ for all $n < 0$, $\mathscr{A}$ the category of commutative DG $\Bbbk$-algebras with homology concentrated in degree 0, which is essentially the same as the category of commutative $\Bbbk$-algebras regarded as complexes concentrated in degree 0, and $\mathbf{P}$ the formally perfect and formally infinitesimally smooth morphisms. A morphism of DGC $\Bbbk$-algebras $A \to B$ satisfies this second condition when for any DGC $\Bbbk$-algebra morphism $f \colon B \to R$ with R in $\mathscr{A}$ and any morphism $g \colon \mathbb{L}_{R/A} \to M$ in $\operatorname{Ho} \operatorname{Mod}(R)$ such that the homology of M is concentrated in positive degrees, the following composite vanishes in $\operatorname{Ho} \operatorname{Mod}(B)$,

$$\mathbb{L}_{B/A} \xrightarrow{f_*} \mathbb{L}_{R/A} \xrightarrow{g} M.$$

(3) $\mathscr{V}$ the category of modules over the Eilenberg–MacLane spectrum $H\Bbbk$ of an arbitrary commutative ring $\Bbbk$ with the positive model structure of [Shi04], $\mathscr{V}_0$ the subcategory of connective $H\Bbbk$-modules M, i.e. $\pi_n M = 0$ for all $n < 0$, $\mathscr{A}$ the category of commutative $H\Bbbk$-algebras with homotopy concentrated in degree 0, which is essentially the same as the category of Eilenberg–MacLane spectra of commutative $\Bbbk$-algebras, and τ and $\mathbf{P}$ as in the previous example. Obviously, we must replace homology groups with homotopy groups in order to obtain the correct definitions in this context.

These three contexts are not general enough, they have special features, e.g. $\mathscr{V}$ is stable. We refer the reader to [TV08] for different kinds of examples. Moreover, (3) extends (2) through the Eilenberg–MacLane functor. The problem with (2) is that it does not work over arbitrary rings, because the category DGC algebras does not admit a transferred model structure, so we must unavoidably pass to the category of spectra, although we may think heuristically of complexes. We are really interested in the last HAG context, we just give the first one as an example of HAG context such that $\mathbf{P}$ contains all formally perfect morphisms. Moreover, (2) and (3) satisfy Artin's conditions, unlike (1), so geometric stacks have a nice infinitesimal theory.

Let $r_X \colon X \xrightarrow{\sim} RX$ and $q_X \colon QX \xrightarrow{\sim} X$ be fibrant and cofibrant replacement functors in $\mathscr{V}$, respectively.

The *dual* of an object Y in $\mathscr{V}$ is $Y^\vee = \mathbb{R}\operatorname{Hom}_{\mathscr{V}}(Y, \mathbb{I})$. For the sake of simplicity, let us assume that Y is fibrant and cofibrant. An object Y is *perfect* if the composition

of the two vertical morphisms in the following diagram is a weak equivalence in $\mathscr{V}$,
(5.2)

$$Y \otimes^{\mathbb{L}} Y^{\vee} \xrightarrow{\cong} \mathrm{Hom}_{\mathscr{V}}(\mathbb{I}, Y) \otimes \mathrm{Hom}_{\mathscr{V}}(Y, R\mathbb{I}) \xrightarrow[q_{\mathbb{I}}^* \otimes \mathrm{id}]{\sim} \mathrm{Hom}_{\mathscr{V}}(Q\mathbb{I}, Y) \otimes \mathrm{Hom}_{\mathscr{V}}(Y, R\mathbb{I})$$

$$\downarrow \otimes$$

$$\mathrm{Hom}_{\mathscr{V}}(Q\mathbb{I} \otimes Y, Y \otimes R\mathbb{I})$$

$$\downarrow (r_{Y \otimes R\mathbb{I}})_*$$

$$\mathbb{R}\mathrm{Hom}_{\mathscr{V}}(Y, Y) \xequal\ \mathrm{Hom}_{\mathscr{V}}(Y, Y) \xrightarrow[\mathrm{Hom}_{\mathscr{V}}(q_{\mathbb{I}} \otimes \mathrm{id}_Y, r_{Y \otimes R\mathbb{I}}(\mathrm{id}_Y \otimes r_Y))]{\sim} \mathrm{Hom}_{\mathscr{V}}(Q\mathbb{I} \otimes Y, R(Y \otimes R\mathbb{I}))$$

In this case, this zig-zag represents an isomorphism $Y \otimes^{\mathbb{L}} Y^{\vee} \cong \mathbb{R}\mathrm{Hom}_{\mathscr{V}}(Y, Y)$ in $\mathrm{Ho}\,\mathscr{V}$. An example of perfect object is the *free* object of rank n,

$$R\left(Q\mathbb{I} \coprod \overset{n}{\cdots} \coprod Q\mathbb{I}\right),$$

i.e. the derived coproduct of n copies of the monoidal unit (n is finite). An object is *projective* if it is a homotopy retract of such a free object. Projective objects are also perfect. A *cellular object* concentrated in the (finite) interval of integers $[a, b]$ is the a-fold suspension of a projective object if $a = b$, and the homotopy cofiber of a morphism $F \to X$ with X a cellular object concentrated in degrees $[a, b-1]$ and F the $(b-1)$-fold suspension of a projective object. We can allow negative values of a and b if $\mathscr{V}$ is stable. Cellular objects concentrated in a finite interval are perfect too. In Example 5.1 (1), (2) and (3) all perfect objects are cellular concentrated in a finite interval.

The representable stack $\mathbb{R}\underline{\mathrm{End}}_{\mathscr{V}}(Y)$ of endomorphisms of Y is given by

$$\mathbb{R}\underline{\mathrm{End}}_{\mathscr{V}}(Y)(A) = \mathrm{Map}_{\mathscr{V}}(Y, Y \otimes A) \simeq \mathrm{Map}_{\mathrm{Mod}(A)}(Y \otimes A, Y \otimes A),$$

and the representable substack $\mathbb{R}\underline{\mathrm{Aut}}_{\mathscr{V}}(Y)$ of automorphisms of Y is defined by the connected components

$$\mathbb{R}\underline{\mathrm{Aut}}_{\mathscr{V}}(Y)(A) \subset \mathbb{R}\underline{\mathrm{End}}_{\mathscr{V}}(Y)(A)$$

of automorphisms of $Y \otimes A$ in $\mathrm{Ho}\,\mathrm{Mod}(A)$ [TV08, Lemma 1.3.7.13 and Proposition 1.3.7.14].

We also consider the stack $\underline{\mathrm{QCoh}}$ of quasi-coherent modules [TV08, Theorem 1.3.7.2],

$$\underline{\mathrm{QCoh}}(A) = |w\mathrm{Mod}(A)_c|,$$

where $\mathrm{Mod}(A)_c \subset \mathrm{Mod}(A)$ denotes the subcategory of cofibrant objects, and the substacks $\underline{\mathrm{Perf}}$ of locally perfect modules [TV08, Definition 1.3.7.5], $\underline{\mathrm{Perf}}_{[a,b]}$ of locally cellular modules concentrated in $[a, b]$, and $\underline{\mathrm{Perf}}_Y$ of modules locally induced by Y. An A-module M is locally perfect (resp. locally cellular concentrated in $[a, b]$, or locally induced by Y) if there is a τ-cover $\{A \to A_i\}_{i \in I}$ with $M \otimes_A^{\mathbb{L}} A_i$ a perfect A_i-module (resp. a cellular A_i-module concentrated in $[a, b]$, or weakly equivalent to $Y \otimes^{\mathbb{L}} A_i$). The simplicial sets $\underline{\mathrm{Perf}}(A)$, $\underline{\mathrm{Perf}}_{[a,b]}(A)$ and $\underline{\mathrm{Perf}}_Y(A)$ consist of the connected components of $|w\mathrm{Mod}(A)_c|$ spanned by the A-modules M satisfying the corresponding property.

In order $\underline{\mathrm{QCoh}}$, $\underline{\mathrm{Perf}}$, $\underline{\mathrm{Perf}}_{[a,b]}$ and $\underline{\mathrm{Perf}}_Y$ to be strictly functorial we should replace $\mathrm{Mod}(A)$ with the equivalent category of quasi-coherent A-modules introduced

in [TV08, §1.3.7], but for the sake of simplicity we will keep the current notation. We will do the same whenever this problem arises.

Proposition 5.3. *The stack* $\underline{\mathrm{Perf}}$ *is categorically locally λ-presentable (i.e. as a simplicial presheaf it takes λ-filtered homotopy colimits in $\mathrm{Comm}(\mathscr{V})$ to λ-filtered homotopy colimits of simplicial sets) and has an affine diagonal. Moreover, if $\mathbf{P}$ contains all formally perfect morphisms (or just perfect if $\lambda = \aleph_0$ and $\mathbb{I}$ is $\aleph_0$-presentable) then $\underline{\mathrm{Perf}}$ is 1-geometric. The same holds for $\underline{\mathrm{Perf}}_{[a,b]}$ and $\underline{\mathrm{Perf}}_Y$, which is the classifying stack of the affine group stack $\mathbb{R}\underline{\mathrm{Aut}}_{\mathscr{V}}(Y)$.*

This result is similar to [TV08, Proposition 2.3.3.1]. The proof is the same, mutatis mutandis. An atlas of $\underline{\mathrm{Perf}}$ is given by

$$\coprod_A \coprod_M \mathbb{R}\underline{\mathrm{Spec}}(A) \longrightarrow \underline{\mathrm{Perf}},$$

where A (resp. M) runs over a set of weak equivalence classes of λ-presentable cofibrant commutative algebras (resp. cofibrant perfect A-modules), and the map $\mathbb{R}\underline{\mathrm{Spec}}(A) \to \underline{\mathrm{Perf}}$ corresponding to a perfect A-module M is represented by the connected component of M in $\underline{\mathrm{Perf}}(A)$. If we restrict M to be locally cellular concentrated in $[a, b]$ or locally induced by Y we obtain atlases for $\underline{\mathrm{Perf}}_{[a,b]}$ and $\underline{\mathrm{Perf}}_Y$, respectively.

The final statement follows from the fact that the homotopy sheaf $\pi_0(\underline{\mathrm{Perf}}_Y)$ is the final sheaf, and the homotopy pull back of $* \to \underline{\mathrm{Perf}}_Y \leftarrow *$ is weakly equivalent to $\mathbb{R}\underline{\mathrm{Aut}}_{\mathscr{V}}(Y)$ by [TV08, Corollary A.0.4]. In particular, the essentially unique morphism $* \to \underline{\mathrm{Perf}}_Y$ is an atlas.

We need weaker hipotheses if Y is free.

Proposition 5.4. *If Y is free and $\mathbf{P}$ contains all formally smooth morphisms (or just smooth if $\lambda = \aleph_0$ and $\mathbb{I}$ is $\aleph_0$-presentable) then $\underline{\mathrm{Perf}}_Y$ is 1-geometric.*

This follows from the fact that, under the hypotheses of this proposition, $\underline{\mathrm{Perf}}_Y$ is the stack of vector bundles of fixed rank which is studied in detail in the last part of [TV08, §1.3.7].

We can weaken the hypothesis on $\mathbf{P}$ if we strengthen the hypotheses on our homotopical algebra context.

Proposition 5.5. *Suppose that for any cellular object Y in $\mathscr{V}$ concentrated in $[-n, 0]$, any commutative algebra R in $\mathscr{A}$, and any R-module M whose underlying object in $\mathscr{V}$ lies in $\mathscr{V}_0$, there are no non-trivial morphisms from Y to ΣM in $\mathrm{Ho}\,\mathscr{V}$ (or equivalently from $\mathbb{I}$ to $Y^\vee \otimes^{\mathbb{L}} \Sigma M$). Assume further that $\mathbf{P}$ contains all morphisms which are formally inifinitesimally smooth and formally perfect. Then $\underline{\mathrm{Perf}}_{[a,b]}$ is 1-geometric. The same holds for $\underline{\mathrm{Perf}}_Y$ if Y is a cellular object concentrated in $[a, b]$.*

The statement of this proposition isolates what is needed to extend the proof of [TV08, Proposition 2.3.5.4] in the context of Example 5.1 (2) to a general context. The proof is again the same, mutatis mutandis. The HAG context in Example 5.1 (3) also satisfies the hypotheses of this proposition.

Let us now turn to our results on moduli spaces of algebras over operads, for which we need the previous results on perfect objects.

Let $\mathcal{O}$ be a weakly cofibrant operad in $\mathscr{V}$. The category $\mathrm{Mod}(A)$ of modules over a commutative algebra A in $\mathscr{V}$ will play the role of $\mathscr{C}$ in previous sections via the

strong symmetric monoidal functor $z = - \otimes A$. The change of coefficients functor $- \otimes_A B \colon \operatorname{Mod}(A) \to \operatorname{Mod}(B)$ induced by a morphism $A \to B$ of commutative algebras in $\mathscr{V}$ preserves $\mathcal{O}$-algebras in these module categories. Hence we can consider the simplicial presheaf $\underline{\operatorname{Alg}}_{\mathscr{V}}(\mathcal{O})$ defined by
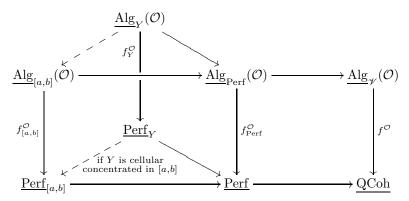
$$\underline{\operatorname{Alg}}_{\mathscr{V}}(\mathcal{O})(A) = |w\operatorname{Alg}_{\operatorname{Mod}(A)}(\mathcal{O})^c|.$$

Here $\operatorname{Alg}_{\operatorname{Mod}(A)}(\mathcal{O})^c \subset \operatorname{Alg}_{\operatorname{Mod}(A)}(\mathcal{O})$ denotes the full subcategory spanned by those $\mathcal{O}$-algebras in $\operatorname{Mod}(A)$ whose underlying A-module is cofibrant.

One can straightforwardly check that $\underline{\operatorname{Alg}}_{\mathscr{V}}(\mathcal{O})$ is a stack along the lines of similar results in [TV08] using the strictification theorem in [TV08, Appendix B]. We call $\underline{\operatorname{Alg}}_{\mathscr{V}}(\mathcal{O})$ the *moduli stack of $\mathcal{O}$-algebras*. This stack comes equipped with a forgetful morphism

$$f^{\mathcal{O}} \colon \underline{\operatorname{Alg}}_{\mathscr{V}}(\mathcal{O}) \longrightarrow \underline{\operatorname{QCoh}}.$$

We also consider the substacks $\underline{\operatorname{Alg}}_{\operatorname{Perf}}(\mathcal{O})$, $\underline{\operatorname{Alg}}_{[a,b]}(\mathcal{O})$ and $\underline{\operatorname{Alg}}_Y(\mathcal{O})$ defined by (homotopy) pulling back,

These are the moduli stacks of $\mathcal{O}$-algebras on locally perfect modules, on locally cellular modules concentrated in $[a,b]$, and on modules locally isomorphic to Y, respectively.

Theorem 5.6. *Suppose $\mathscr{V}$ has a set of generating cofibrations with cofibrant sources. Let $\mathcal{O}$ be a weakly cofibrant operad in $\mathscr{V}$ and Y a perfect object. The homotopy pull-back of $f^{\mathcal{O}}_{\operatorname{Perf}}$ along any morphism $\mathbb{R}\underline{\operatorname{Spec}}(A) \to \underline{\operatorname{Alg}}_{\operatorname{Perf}}(\mathcal{O})$ is representable. In particular, $\underline{\operatorname{Alg}}_{\operatorname{Perf}}(\mathcal{O})$, $\underline{\operatorname{Alg}}_{[a,b]}(\mathcal{O})$ and $\underline{\operatorname{Alg}}_Y(\mathcal{O})$ are 1-geometric if $\mathbf{P}$ contains all formally perfect morphisms (or just perfect if $\lambda = \aleph_0$ and $\mathbb{I}$ is $\aleph_0$-presentable). Moreover, $\underline{\operatorname{Alg}}_{[a,b]}(\mathcal{O})$ and $\underline{\operatorname{Alg}}_Y(\mathcal{O})$ are 1-geometric if $\mathbf{P}$ contains all morphisms which are formally infinitesimally smooth and formally perfect and Y is cellular concentrated in a finite interval.*

Proof. By Theorem 3.11 and Corollary 3.13, we can suppose without loss of generality that $\mathcal{O}$ is cofibrant, and moreover an $\mathcal{F}(I_{\mathbb{N}})$-cell complex in the sense of the proof of Lemma 3.12. Hence this theorem is a consequence of the following two lemmas. $\square$

The cofibrancy condition on the sources of generating cofibrations added to $\mathscr{V}$ in the statement of Theorem 5.6 makes $\mathscr{V}$ a *tractable model category* in the sense of [Bar10]. We do not know a single example of a left proper combinatorial model

category that is not tractable, therefore the extra hipothesis on $\mathscr{V}$ does not seem to be restrictive.

The following lemma is a special case of Theorem 5.6. It can be applied to the initial operad, which is free on the initial object.

Lemma 5.7. *If $\mathcal{O}$ is a free operad on a cofibrant object then the homotopy pull-back of $f_{\mathrm{Perf}}^{\mathcal{O}}$ along any morphism $g \colon \mathbb{R}\underline{\mathrm{Spec}}(A) \to \underline{\mathrm{Alg}}_{\mathrm{Perf}}(\mathcal{O})$ is representable.*

Proof. The morphism g is classified by a locally perfect A-module M, that we can suppose to be fibrant and cofibrant.

Suppose $\mathcal{O} = \mathcal{F}(U)$ for some $U = \{U_n\}_{n \geq 0}$ cofibrant in $\mathscr{V}^{\mathbb{N}}$, i.e. each U_n is cofibrant in $\mathscr{V}$. We define the stack $\underline{\mathrm{Map}}_{\mathrm{Op}(\mathscr{V})}(\mathcal{F}(U), \mathbf{End}_{\mathrm{Mod}(A)}(M))$ of $\mathcal{F}(U)$-algebra structures on M as

$$\underline{\mathrm{Map}}_{\mathrm{Op}(\mathscr{V})}(\mathcal{F}(U), \mathbf{End}_{\mathrm{Mod}(A)}(M))(B) = \prod_{n \geq 0} \mathrm{Map}_{\mathrm{Mod}(A)}(U_n \otimes M^{\otimes_A n}, M \otimes_A B)$$

for any commutative A-algebra B, i.e. this is a stack in the induced HAG context over $\mathrm{Mod}(A)$. This stack is represented by the free commutative A-algebra C generated by

$$\coprod_{n \geq 0} U_n \otimes M^{\otimes_A n} \otimes_A M^{\vee},$$

where $M^{\vee}$ denotes the dual of M in $\mathrm{Mod}(A)$.

Notice that, by adjunction, $\underline{\mathrm{Map}}_{\mathrm{Op}(\mathscr{V})}(\mathcal{F}(U), \mathbf{End}_{\mathrm{Mod}(A)}(M))(B)$ is weakly equivalent to

$$K = \mathrm{Map}_{\mathrm{Op}(\mathscr{V})}(\mathcal{F}(U), \mathbf{End}_{\mathrm{Mod}(B)}(\widetilde{M \otimes_A B})),$$

where $\widetilde{M \otimes_A B}$ denotes a fibrant replacement of $M \otimes_A B$ in the category of B-modules, but we cannot define the stack $\underline{\mathrm{Map}}_{\mathrm{Op}(\mathscr{V})}(\mathcal{F}(U), \mathbf{End}_{\mathrm{Mod}(A)}(M))$ in this way since K is not functorial in B.

Consider the morphism

$$h \colon \mathbb{R}\underline{\mathrm{Spec}}(C) \longrightarrow \underline{\mathrm{Alg}}_{\mathrm{Perf}}(\mathcal{F}(U))$$

represented by the vertex in $\underline{\mathrm{Alg}}_{\mathrm{Perf}}(\mathcal{F}(U))(C) \subset |w\,\mathrm{Alg}_{\mathrm{Mod}(C)}(\mathcal{F}(U))^c|$ determined by the fibrant-cofibrant C-module $\widetilde{M \otimes_A C}$ with the $\mathcal{F}(U)$-algebra structure defined as follows. Let $r_X \colon X \overset{\sim}{\rightarrowtail} RX$ and $q_X \colon QX \overset{\sim}{\twoheadrightarrow} X$ be now fibrant and cofibrant

replacement functors in $\text{Mod}(A)$, respectively. We look at the zig-zag in $\text{Mod}(A)$

$$U_n \otimes M^{\otimes_A n} \cong U_n \otimes M^{\otimes_A n} \otimes_A A$$

$$\downarrow \text{id}_{\left(U_n \otimes M^{\otimes_A n}\right)} \otimes_A \text{(identity in } M)$$

$$U_n \otimes M^{\otimes_A n} \otimes_A \text{Hom}_{\text{Mod}(A)}(M, M)$$

$$\downarrow \sim$$

$$U_n \otimes M^{\otimes_A n} \otimes_A \text{Hom}_{\text{Mod}(A)}(QA \otimes_A M, R(M \otimes_A RA)) \quad \Big\} \, \text{id}_{\left(U_n \otimes M^{\otimes_A n}\right)} \otimes_A (5.2)$$

$$\uparrow \sim$$

$$(U_n \otimes M^{\otimes_A n}) \otimes_A M \otimes_A M^\vee$$

$$\cong \, \Big\downarrow \text{shuffle} \otimes_A \text{id}_{M^\vee}$$

$$M \otimes_A (U_n \otimes M^{\otimes_A n}) \otimes_A M^\vee$$

$$\downarrow \text{id}_M \otimes_A \text{inclusion}$$

$$M \otimes_A C$$

$$\downarrow \sim$$

$$\widetilde{M \otimes_A} C.$$

Here, the morphism 'inclusion' is the restriction of the unit of the free A-algebra adjunction. We choose a morphism

$$\tilde{\nu}_n \colon U_n \otimes M^{\otimes_A n} \longrightarrow \widetilde{M \otimes_A} C$$

representing this zig-zag in $\text{Ho}\,\text{Mod}(A)$. If we apply the lifting axiom to the following commutative square in $\text{Mod}(C)$,

$$
\begin{array}{ccc}
U_n \otimes (M \otimes_A C)^{\otimes_C n} & \xrightarrow{\text{adjoint of } \tilde{\nu}_n} & \widetilde{M \otimes_A} C \\
\sim \Big\downarrow & & \Big\downarrow \\
U_n \otimes (\widetilde{M \otimes_A} C)^{\otimes_C n} & \longrightarrow & 0
\end{array}
$$

where 0 denotes the zero C-module, we obtain a morphism

$$\nu_n \colon U_n \otimes (\widetilde{M \otimes_A} C)^{\otimes_C n} \longrightarrow \widetilde{M \otimes_A} C.$$

These morphisms ν_n define the desired $\mathcal{F}(U)$-algebra structure on $\widetilde{M \otimes_A} C$.

Denote $l \colon A \to C$ the unit of C. The following diagram of stacks

$$(5.8) \qquad
\begin{array}{ccc}
\mathbb{R}\underline{\text{Spec}}(C) & \xrightarrow{\;l^*\;} & \mathbb{R}\underline{\text{Spec}}(A) \\
h \Big\downarrow & \Downarrow \alpha & \Big\downarrow g \\
\underline{\text{Alg}}_{\text{Perf}}(\mathcal{F}(U)) & \xrightarrow[f_{\text{Perf}}^{\mathcal{F}(U)}]{} & \underline{\text{Perf}}
\end{array}
$$

commutes up to the homotopy $\alpha\colon gl^* \Rightarrow f_{\mathrm{Perf}}^{\mathcal{F}(U)} h$ defined by the edge (i.e. 1-simplex) in $\underline{\mathrm{Perf}}(C) \subset |w\mathrm{Mod}(C)_c|$ given by the trivial cofibration $M \otimes_A C \overset{\sim}{\rightarrowtail} \widetilde{M \otimes_A C}$.

In order to finish the proof, it is enough to notice that (5.8) is a homotopy pull-back. This follows from the fact that for any A-algebra B, diagram (5.8) induces a weak equivalence between the homotopy fiber of $l^*(B)$ at $A \to B$ and the homotopy fiber of $f_{\mathrm{Perf}}^{\mathcal{F}(U)}(B)$ at $M \otimes_A B$. Indeed, the homotopy fiber of $l^*(B)$ at $A \to B$ is weakly equivalent to K since C represents the stack of $\mathcal{F}(U)$-algebra structures on M, and the homotopy fiber of $f_{\mathrm{Perf}}^{\mathcal{F}(U)}(B)$ at $M \otimes_A B$ is weakly equivalent to K by Theorem 4.6. It is a tedious but straightforward task to check that the map induced by (5.8) between these homotopy fibers can be identified with the identity in K through these weak equivalences, hence we are done. $\qquad\square$

The next lemma allows the proof of Theorem 5.6 by induction.

Lemma 5.9. *The homotopy pull-back of $f_{\mathrm{Perf}}^-\colon \underline{\mathrm{Alg}}_{\mathrm{Perf}}(-) \to \underline{\mathrm{Perf}}$ along any morphism $\mathbb{R}\underline{\mathrm{Spec}}(A) \to \underline{\mathrm{Perf}}$ takes homotopy colimits of weakly cofibrant operads to homotopy limits of stacks. In particular, it takes homotopy push-outs of weakly cofibrant operads to homotopy pull-backs of stacks.*

Proof. Mapping spaces are known to take homotopy colimits in the first variable to homotopy limits of simplicial sets. This fact and Theorem 4.6 prove the claim. $\quad\square$

Corollary 5.10. *The functor $\underline{\mathrm{Alg}}_{\mathrm{Perf}}(-)$ takes homotopy colimits of weakly cofibrant operads to homotopy limits of stacks. In particular, it takes homotopy push-outs of weakly cofibrant operads to homotopy pull-backs of stacks.*

Given a locally perfect A-module M classified by $g\colon \mathbb{R}\underline{\mathrm{Spec}}(A) \to \underline{\mathrm{Perf}}$ we define the representable *moduli stack of $\mathcal{O}$-algebra structures on M* as the homotopy pull-back of $f_{\mathrm{Perf}}^{\mathcal{O}}$ along g, which is the same as the homotopy pull-back of $f_{[a,b]}^{\mathcal{O}}$ along g if M is locally cellular concentrated in $[a,b]$, and we denote it by

$$\underline{\mathrm{Map}}_{\mathrm{Op}(\mathscr{V})}(\mathcal{O}, \mathtt{End}_{\mathrm{Mod}(A)}(M)).$$

This name is justified by Theorem 4.6, and it is compatible with the terminology in the proof of Lemma 5.7.

Let Y be a perfect object in $\mathscr{V}$. Since $* \to \underline{\mathrm{Perf}}_Y$ is an atlas, the canonical map

$$\underline{\mathrm{Map}}_{\mathrm{Op}(\mathscr{V})}(\mathcal{O}, \mathtt{End}_{\mathscr{V}}(Y)) \longrightarrow \underline{\mathrm{Alg}}_Y(\mathcal{O})$$

is also an atlas, and we can regard $\underline{\mathrm{Alg}}_Y(\mathcal{O})$ as a quotient of $\underline{\mathrm{Map}}_{\mathrm{Op}(\mathscr{V})}(\mathcal{O}, \mathtt{End}_{\mathscr{V}}(Y))$ by the action of the representable group stack $\mathbb{R}\underline{\mathrm{Aut}}_{\mathscr{V}}(Y)$ classified by $\underline{\mathrm{Perf}}_Y$, as in the classical case sketched in the introduction.

The following definition is needed for the computation of tangent complexes of the moduli stack of $\mathcal{O}$-algebras on locally cellular objects concentrated in a finite interval.

Definition 5.11. Let $\mathcal{O}$ be an operad and A a commutative algebra in $\mathscr{V}$. The *space of derivations* of an $\mathcal{O}$-algebra B in $\mathrm{Mod}(A)$ with values in a B-module M is the space $\mathbb{R}\mathrm{Der}_A^{\mathcal{O}}(B, M)$ defined as the homotopy fiber of

$$(B \ltimes M \twoheadrightarrow B)_*\colon \mathrm{Map}_{\mathrm{Alg}_{\mathrm{Mod}(A)}(\mathcal{O})}(B, B \ltimes M) \longrightarrow \mathrm{Map}_{\mathrm{Alg}_{\mathrm{Mod}(A)}(\mathcal{O})}(B, B)$$

at the identity in B. One can arrange the construction of $\mathbb{R}\mathrm{Der}_A^{\mathcal{O}}(B, M)$ to be functorial in M.

The *module of derivations* of B in M is the object $\mathbb{R}\underline{\mathrm{Der}}_A^{\mathcal{O}}(B, M)$ in $\mathrm{Mod}(A)$ representing the simplicial presheaf

$$\mathrm{Mod}(A)^{\mathrm{op}} \longrightarrow \mathbf{Set}^{\Delta^{\mathrm{op}}},$$

$$N \mapsto \mathbb{R}\mathrm{Der}_A^{\mathcal{O}}(B, M \otimes_A N^{\vee}),$$

i.e. this simplicial presheaf is weakly equivalent to $\mathrm{Map}_{\mathrm{Mod}(A)}(-, \mathbb{R}\underline{\mathrm{Der}}_A^{\mathcal{O}}(B, M))$, with the model structure on simplicial presheaves where weak equivalences and fibrations are defined pointwise.

Proposition 5.12. *Under the hypotheses of Theorem 5.6, if we assume in addition that the suspension functor in $\mathrm{Ho}\,\mathrm{Mod}(A)$ is fully faithful, then the tangent complex of the stack $\underline{\mathrm{Alg}}_{\mathrm{Perf}}(\mathcal{O})$ at a point $\mathbb{R}\underline{\mathrm{Spec}}A \to \underline{\mathrm{Alg}}_{\mathrm{Perf}}(\mathcal{O})$ classified by an $\mathcal{O}$-algebra B in $\mathrm{Mod}(A)$ is*

$$\mathbb{T}_{\underline{\mathrm{Alg}}_Y(\mathcal{O}),B} \simeq \Sigma\mathbb{R}\underline{\mathrm{Der}}_A^{\mathcal{O}}(B, B).$$

The equivalence in the statement takes place in the stabilization of $\mathrm{Mod}(A)$, in the sense of [Hov01], if $\mathrm{Mod}(A)$ is not stable. The proof reduces to showing that both objects represent the same presheaf in the stabilization of $\mathrm{Mod}(A)_0 = \mathscr{V}_0 \cap \mathrm{Mod}(A)$. An abstraction of the argument in [TV08, Proposition 2.2.6.9] works with minor modifications, compare also [TV08, Corollary 2.3.5.9]. We do not reproduce this argument here.

Let $\mathtt{Ass}$ and $\mathtt{uAss}$ be the operads for associative and unital associative algebras in $\mathscr{V}$, respectively. These operads are given by

$$\mathtt{Ass}(0) = 0 \text{ the zero object}, \quad \mathtt{uAss}(0) = \mathbb{I}, \quad \mathtt{Ass}(n) = \mathtt{uAss}(n) = \mathbb{I}, \quad n > 0,$$

and all composition laws are the obvious morphisms (either trivial or isomorphisms). The unique morphism

$$\psi \colon \mathtt{Ass} \to \mathtt{uAss}$$

is a model for the map forgetting the unit from unital associative algebras to associative algebras. Let

$$\tilde{\psi} \colon \mathtt{A}_{\infty} \to \mathtt{uA}_{\infty}$$

be a cofibrant replacement of ψ in $\mathrm{Op}(\mathscr{V})$.

Proposition 5.13. *In the HAG contexts of Example 5.1 (2) and (3), given a commutative algebra A in $\mathscr{V}$ and a locally perfect A-module M the morphism of affine stacks*

$$\tilde{\psi}^* \colon \underline{\mathrm{Map}}_{\mathrm{Op}(\mathscr{V})}(\mathtt{uA}_{\infty}, \mathtt{End}_{\mathrm{Mod}(A)}(M)) \longrightarrow \underline{\mathrm{Map}}_{\mathrm{Op}(\mathscr{V})}(\mathtt{A}_{\infty}, \mathtt{End}_{\mathrm{Mod}(A)}(M))$$

is a formal Zariski open immersion.

Proof. Example 5.1 (2) is extended by (3), so it is enough to consider this second context. Since $\tilde{\psi}^*$ is a morphism between representable stacks over $\mathbb{R}\underline{\mathrm{Spec}}(A)$, we can take a morphism of commutative A-algebras,

$$\bar{\psi} \colon A_{\infty} \longrightarrow uA_{\infty},$$

representing $\tilde{\psi}^*$.

By [TV08, Remark 1.2.6.2 and Corollary 1.2.6.6] we must show that $\bar{\psi}$ is a homotopy epimorphism of A-algebras in the sense of [Mur11b, Definition 2.1], i.e. we must prove that for any commutative A-algebra B, the morphism

$$\bar{\psi}^* \colon \mathrm{Map}_{\mathrm{Comm}(\mathrm{Mod}(A))}(uA_{\infty}, B) \longrightarrow \mathrm{Map}_{\mathrm{Comm}(\mathrm{Mod}(A))}(A_{\infty}, B)$$

induces an injection on π_0 and isomorphisms in all higher homotopy groups π_n, $n \geq 1$, with all possible base points.

By Theorem 4.6, the map $\bar{\psi}^*$ is weakly equivalent to

$$\mathrm{Map}_{\mathrm{Op}(\mathscr{V})}(\mathtt{uA}_\infty, \mathrm{End}_{\mathrm{Mod}(B)}(\widetilde{M \otimes_A B})) \xrightarrow{\tilde{\psi}^*} \mathrm{Map}_{\mathrm{Op}(\mathscr{V})}(\mathtt{A}_\infty, \mathrm{End}_{\mathrm{Mod}(B)}(\widetilde{M \otimes_A B})).$$

Here, $\widetilde{M \otimes_A B}$ denotes a fibrant-cofibrant replacement of the B-module $M \otimes_A B$.

The Quillen equivalence between $H\Bbbk$-modules and DG-modules over $\Bbbk$ [Shi07] induces a Quillen equivalence between their categories of operads. Therefore, by [Mur11b, Theorem 2.3], $\tilde{\psi}^*$ induces an injection on sets of connected components and isomorphisms in all higher homotopy groups, hence so does $\bar{\psi}^*$. $\square$

The following result is a consequence of [TV08, Proposition 1.3.6.3 (2)].

Corollary 5.14. *In the HAG contexts of Example 5.1 (2) and (3), the morphism*

$$\tilde{\psi}^* \colon \underline{\mathrm{Alg}}_{[a,b]}(\mathtt{uA}_\infty) \longrightarrow \underline{\mathrm{Alg}}_{[a,b]}(\mathtt{A}_\infty)$$

is a formal Zariski open immersion for any pair of integers $a \leq b$. So it is the morphism

$$\tilde{\psi}^* \colon \underline{\mathrm{Alg}}_Y(\mathtt{uA}_\infty) \longrightarrow \underline{\mathrm{Alg}}_Y(\mathtt{A}_\infty)$$

for any perfect object Y in $\mathscr{V}$.

Remark 5.15. This remark is about Example 5.1 (3), but for most of the discussion we can take $\mathscr{V}$ to be the category of DG $\Bbbk$-modules. The model categories of (associative but not necessarily unital) DG $\Bbbk$-algebras and $\mathtt{A}_\infty$-algebras are Quillen equivalent. The bar and cobar constructions define an adjoint pair [LH03, Lemme 1.2.2.5]

$$\mathrm{Alg}_{\mathscr{V}}(\mathtt{Ass}) = \text{DG-algebras} \underset{\Omega}{\overset{B}{\rightleftarrows}} \text{DG-coalgebras}$$

such that $\Omega B(A)$ is cofibrant if A is a DG-algebra with underlying bounded complex of finitely generated projective $\Bbbk$-modules. In order to compute $\mathrm{Map}_{\mathrm{Alg}_{\mathscr{V}}(\mathtt{Ass})}(A, -)$ we can tensor the DG-coalgebra $B(A)$ with the cosimplicial DG-coalgebra $C_*(\Delta^\bullet, \Bbbk)$ of normalized chains of standard simplicies with coefficients in $\Bbbk$ and then take the cobar construction,

$$\mathrm{Map}_{\mathrm{Alg}_{\mathscr{V}}(\mathtt{Ass})}(A, -) = \mathrm{Hom}_{\mathrm{Alg}_{\mathscr{V}}(\mathtt{Ass})}(\Omega(B(A) \otimes C_*(\Delta^\bullet, \Bbbk)), -).$$

Such a trick also applies to the computation of $\mathrm{Map}_{\mathscr{V}}(N, -)$ for N a cofibrant object in $\mathscr{V}$,

$$\mathrm{Map}_{\mathscr{V}}(N, -) = \mathrm{Hom}_{\mathscr{V}}(N \otimes C_*(\Delta^\bullet, \Bbbk), -).$$

This can be used to compute $\mathbb{R}\underline{\mathrm{Der}}_{\Bbbk}^{\mathtt{Ass}}(A, M)$, for M any A-bimodule, from the very definition. It is a perturbation of the DG-module

(5.16)
$$\bigoplus_{n \geq 1} \Sigma^{1-n} \mathrm{Hom}_{\mathscr{V}}(A^{\otimes n}, M)$$

by the Hochschild differential,

$$\delta \colon \mathrm{Hom}_{\mathscr{V}}(A^{\otimes n}, M) \longrightarrow \mathrm{Hom}_{\mathscr{V}}(A^{\otimes(n+1)}, M), \qquad n \geq 1,$$

$$\delta(\varphi)(a_1, \ldots, a_{n+1}) = a_1 \varphi(a_2, \ldots, a_{n+1}) + \sum_{i=1}^n (-1)^i \varphi(a_1, \ldots, a_i a_{i+1}, \ldots, a_{n+1})$$

$$+ (-1)^{n+1} \varphi(a_1, \ldots, a_n) a_{n+1}.$$

The *Hochschild complex*, whose homology (up to passing to cohomological grading and shifting by 1) is the Hochschild cohomology $HH^*(A, M)$ of A with coefficients in M, is defined in the same way but adding an extra direct summand for $n = 0$ to (5.16). In particular we get a long exact sequence,

$$\cdots \to H_n(M) \to H_n \mathbb{R}\underline{\mathrm{Der}}_{\Bbbk}^{\mathtt{Ass}}(A, M) \to HH^{1-n}(A, M) \to H_{n-1}(M) \to \cdots.$$

The aforementioned Quillen equivalence identifies, up to quasi-isomorphism, the DG $\Bbbk$-modules $\mathbb{R}\underline{\mathrm{Der}}_{\Bbbk}^{\mathtt{A}\infty}(A, M)$ and $\mathbb{R}\underline{\mathrm{Der}}_{\Bbbk}^{\mathtt{Ass}}(A, M)$. Moreover, Corollary 5.14 implies that if A is unital then

$$\mathbb{T}_{\underline{\mathrm{Alg}}_Y(\mathtt{uA}_\infty), A} \simeq \mathbb{T}_{\mathrm{Alg}_Y(\mathtt{A}_\infty), A},$$

therefore, by Proposition 5.13,

$$\mathbb{R}\underline{\mathrm{Der}}_{\Bbbk}^{\mathtt{uA}\infty}(A, A) \simeq \mathbb{R}\underline{\mathrm{Der}}_{\Bbbk}^{\mathtt{A}\infty}(A, A) \simeq \mathbb{R}\underline{\mathrm{Der}}_{\Bbbk}^{\mathtt{Ass}}(A, A).$$

This computation says that the 'right' cohomology theory for unital DG-algebras is still Hochschild cohomology, despite it seems not to take into account the existence of a unit, compare [HM10, 6.6.7].

There is also a cobar construction for $\mathtt{A}_\infty$-algebras producing a DG-coalgebra [LH03, Définition 1.2.2.3]. This construction can be applied to the computation of these DG-modules of derivations, with the only difference that the Hochschild differential becomes more complicated [LH03, §B.4].

If instead of a DG or $\mathtt{A}_\infty$ $\Bbbk$-algebra we had taken an $\mathtt{A}_\infty$-algebra over an $H\Bbbk$-algebra, then we should replace the Hochschild complex by the topological Hochschild cohomology spectrum, compare [Laz01].

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